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A GENERALIZED SKEW LOGISTIC DISTRIBUTION

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Abstract:

- In this paper, we introduce a generalized skew logistic distribution that contains the usual skew logistic distribution as a special case. Several mathematical properties of the distribution are discussed like the cumulative distribution function and moments. Furthermore, estimation using the method of maximum likelihood and the Fisher information matrix are investigated. Two real data applications illustrate the performance of the distribution.

Key-Words:

- *estimation; logistic distribution; moments.*

AMS Subject Classification:

- 62E15.

1. INTRODUCTION

Azzalini [2] introduced the skew normal distribution specified by the probability density function (pdf):

$$f_{SN}(x;\lambda) = 2\,\phi(x)\,\Phi(\lambda x)\,, \qquad -\infty < x < \infty\,, \tag{1.1}$$

where $\lambda \in R$ is the skewness parameter, $\phi(x)$ is the standard normal pdf, and $\Phi(x)$ is the standard normal cumulative distribution function (cdf). Although, Azzalini introduced the skew version (1.1) for the normal distribution, this idea can be applied to any symmetric pdf. Along the same line, the skew logistic distribution with the skewness parameter λ can be proposed as follows. Consider the standard logistic distribution specified by the cdf

$$H(x) = \frac{1}{1+\exp(-x)}\,, \qquad -\infty < x < \infty\,,$$

and the pdf

$$h(x) = \frac{\exp(-x)}{\bigl(1+\exp(-x)\bigr)^2}\,, \qquad -\infty < x < \infty\,.$$

Using the idea of Azzalini [2], the pdf of the usual skew logistic distribution is given by

$$f_{SL}(x;\lambda) = 2\,h(x)\,H(\lambda x) = \frac{2\exp(-x)}{\bigl(1+\exp(-x)\bigr)^2\bigl(1+\exp(-\lambda x)\bigr)} \tag{1.2}$$

for $-\infty < x < \infty$ and $\lambda \in R$. The properties of this distribution have been studied extensively in the literature. See, for example, Nadarajah [12] and Gupta and Kundu [9]. The skew logistic distribution in (1.2) has also received applications; for example, Koessler and Kumar [11] illustrate an application with respect to an adaptive test for the two-sample scale problem based on U-statistics.

Because of the increasing popularity of (1.2), one would like to have generalizations that are more flexible. The aim of this paper is to construct a new generalization of (1.2) using the type III generalized logistic distribution instead of the standard logistic distribution. We study mathematical properties of this new generalization and discuss real data applications.

The type III generalized logistic distribution has the pdf (see Johnson *et al.* [10])

$$g_\alpha(x) = \frac{1}{B(\alpha,\alpha)} \frac{\exp(-\alpha x)}{\big(1+\exp(-x)\big)^{2\alpha}}$$

for $-\infty < x < \infty$ and $\alpha > 0$. This distribution is symmetric for every α. When $\alpha = 1$, the above pdf reduces to the standard logistic pdf. This distribution has the cdf

$$G_\alpha(x) = \frac{B_y(\alpha,\alpha)}{B(\alpha,\alpha)} \ ,$$

where $y = \big(1+\exp(-x)\big)^{-1}$ and

$$B(\alpha,\alpha) = \frac{\{\Gamma(\alpha)\}^2}{\Gamma(2\alpha)} \ .$$

Here,

$$\Gamma(a) = \int_0^\infty t^{a-1}\exp(-t)\,dt \ , \qquad B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$$

are the gamma function and the incomplete beta function, respectively.

Now, we define the new skew logistic distribution as follows. If a random variable X has the following pdf

$$\text{(1.3)} \quad f(x;\alpha,\lambda) = 2\,g_\alpha(x)\,G_\alpha(\lambda x) \ , \qquad -\infty < x < \infty, \quad \alpha > 0, \quad \lambda \in R \ ,$$

then we say that X has a general skew logistic (GSL) distribution. We write $X \sim \mathrm{GSL}(\alpha,\lambda)$.

From (1.3), some basic properties of $\mathrm{GSL}(\alpha,\lambda)$ can be noted as follows:

- **(i)** When $\alpha = 1$, (1.3) reduces to the usual skew logistic pdf;
- **(ii)** When $\lambda = 0$, (1.3) reduces to the type III generalized logistic pdf;
- **(iii)** If $X \sim \mathrm{GSL}(\alpha,\lambda)$, then $-X \sim \mathrm{GSL}(\alpha,-\lambda)$;
- **(iv)** $f(x;\alpha,\lambda) + f(x;\alpha,-\lambda) = 2g_\alpha(x)$ for all $x \in R$;
- **(v)** $f(x;\alpha,\lambda) \to 2g_\alpha(x)\,I\{x \geq 0\}$ as $\lambda \to +\infty$ and $f(x;\alpha,\lambda) \to 2g_\alpha(x)$ $I\{x \leq 0\}$ as $\lambda \to -\infty$ for all α;
- **(vi)** $f(x;\alpha,\lambda) \to 0$ as $x \to \pm\infty$ for all $\alpha > 0$ and $\lambda \in R$.

Numerical investigations show that (1.3) has a single mode. The mode is at x_0, where x_0 is the root of

$$\frac{\lambda}{\alpha}\,\frac{g_\alpha(x)}{G_\alpha(x)} - \frac{1-\exp(-x)}{1+\exp(-x)} = 0 \ .$$

Figures 1 and 2 illustrate possible shapes of the pdf (1.3) for $\alpha = 2$ and selected values of λ.

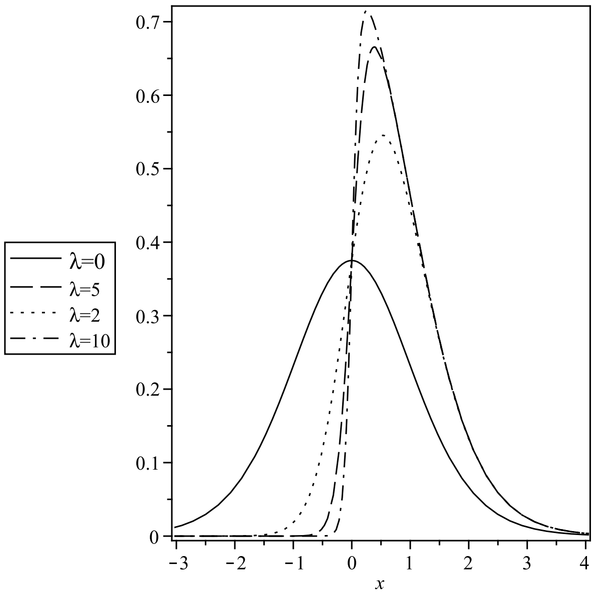

Figure 1: Plots of GSL(α, λ) pdf for $\alpha = 2$ and $\lambda > 0$.

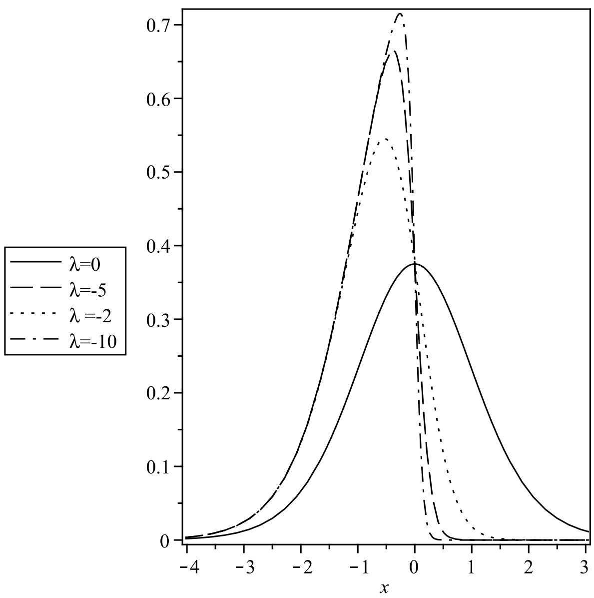

Figure 2: Plots of GSL(α, λ) pdf for $\alpha = 2$ and $\lambda < 0$.

Simulation from (1.3) is straight-forward by using the following representation due to Azzalini [3]:

- $X = S_U U$, where, conditionally on $U = u$, $S_U = +1$ with probability $G_\alpha(\lambda u)$ and $S_U = -1$ with probability $1 - G_\alpha(\lambda u)$;
- $X = S_U |U|$, where, conditionally on $|U| = |u|$, $S_U = +1$ with probability $G_\alpha(\lambda |u|)$ and $S_U = -1$ with probability $1 - G_\alpha(\lambda |u|)$.

Both these representations have physical meanings as explained in Azzalini [3].

In the sequel, we shall use the following functions:

$$\tau_1(b,q) = \sum_{j=0}^{\infty} \frac{\binom{-2q}{j}}{\left(q+j\right)^b}\,, \qquad \tau_2(a,b,q,\lambda) = \sum_{j=0}^{\infty} \frac{\binom{-2q}{j}}{\left(\lambda b+q+j\right)^a}$$

and, the Gauss hypergeometric function defined by

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k}\,\frac{x^k}{k!}\,,$$

where $(z)_k = z(z+1)\cdots(z+k-1)$ denotes the ascending factorial.

Throughout the rest of this paper (unless otherwise stated), we shall assume that $\lambda > 0$ since the corresponding results for $\lambda < 0$ can be obtained using the fact that $-X$ has the pdf $2\,g_\alpha(x)\,G_\alpha(-\lambda x)$.

Some results of this paper require certain series representations of the general skew logistic pdf (1.3), which we derive now. Using the Taylor series expansion for $[1+\exp(-\lambda x)]^{-1}$, we can obtain

$$G_\alpha(\lambda x) = \begin{cases} \dfrac{1}{B(\alpha,\alpha)} \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\alpha-1}{i}\binom{-(\alpha+i)}{j} \\ \qquad \cdot\, \dfrac{(-1)^i \exp(-j\lambda x)}{i+\alpha}\,, & x > 0\,, \\ \dfrac{1}{B(\alpha,\alpha)} \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \binom{\alpha-1}{i}\binom{-(\alpha+i)}{j} \\ \qquad \cdot\, \dfrac{(-1)^i \exp\big(\lambda x(i+\alpha+j)\big)}{i+\alpha}\,, & x < 0\,. \end{cases}$$

Substituting this into (1.3), a double series representation for the general skew logistic pdf can be obtained as

$$(1.4)\qquad f(x;\alpha,\lambda) = \begin{cases} \dfrac{2}{B(\alpha,\alpha)^2\big(1+\exp(-x)\big)^{2\alpha}} \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j} \\ \qquad\cdot\dfrac{(-1)^i\exp\big(-x(\lambda j+\alpha)\big)}{i+\alpha}\,, & x>0\,, \\ \dfrac{2}{B(\alpha,\alpha)^2\big(1+\exp(-x)\big)^{2\alpha}} \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j} \\ \qquad\cdot\dfrac{(-1)^i\exp\big(x(\lambda i+\lambda\alpha+\lambda j-\alpha)\big)}{i+\alpha}\,, & x<0\,. \end{cases}$$

By expanding the terms in the denominators of (1.4), one can also obtain the triple series representation

$$(1.5)\qquad f(x;\alpha,\lambda) = \begin{cases} \dfrac{2}{B(\alpha,\alpha)^2} \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}\binom{-2\alpha}{k} \\ \qquad\cdot\dfrac{(-1)^i\exp\big(-x(\lambda j+k+\alpha)\big)}{i+\alpha}\,, & x>0\,, \\ \dfrac{2}{B(\alpha,\alpha)^2} \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}\binom{-2\alpha}{k} \\ \qquad\cdot\dfrac{(-1)^i\exp\Big(x\big(\lambda(i+\alpha+j)+k+\alpha\big)\Big)}{i+\alpha}\,, & x<0\,. \end{cases}$$

2. CUMULATIVE DISTRIBUTION FUNCTION

Using the double and triple series representations in (1.4) and (1.5), we derive some formulas for the cdf corresponding to (1.3). First, we use the double series representation in (1.4). If $x > 0$, then the cdf $F(x)$ can be written as

$$(2.1)\qquad F(x) = F(0) + \int_0^x \frac{2}{B(\alpha,\alpha)^2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}(-1)^i}{(i+\alpha)\big(1+\exp(-t)\big)^{2\alpha}} \cdot\exp\big(-t(\lambda j+\alpha)\big)\,dt =$$

$$
\begin{aligned}
(2.2)\qquad &= F(0) + \frac{2}{B(\alpha,\alpha)^2} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}(-1)^i}{i+\alpha} \\
&\quad \cdot \int_0^x \frac{\exp\bigl(-t(\lambda j+\alpha)\bigr)}{\bigl(1+\exp(-t)\bigr)^{2\alpha}}\, dt \\
(2.3)\qquad &= F(0) + \frac{2}{B(\alpha,\alpha)^2} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}(-1)^i}{i+\alpha} \\
&\quad \cdot \int_{\exp(-x)}^1 \frac{z^{\lambda j+\alpha-1}}{(1+z)^{2\alpha}}\, dz \\
(2.4)\qquad &= F(0) + \frac{2}{B(\alpha,\alpha)^2} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}(-1)^i}{i+\alpha}\, I(x)\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
I(x) &= \int_{\exp(-x)}^1 \frac{z^{\lambda j+\alpha-1}}{(1+z)^{2\alpha}}\, dz = \int_0^1 \frac{z^{\lambda j+\alpha-1}}{(1+z)^{2\alpha}}\, dz - \int_0^{\exp(-x)} \frac{z^{\lambda j+\alpha-1}}{(1+z)^{2\alpha}}\, dz \\
&= I_1 - I_2\ .
\end{aligned}
$$

By equation (3.194.1) in Gradshteyn and Ryzhik [8], the integrals I_1 and I_2 can be calculated as

$$
(2.5)\qquad I_1 = \int_0^1 \frac{z^{\lambda j+\alpha-1}}{(1+z)^{2\alpha}}\, dz = \frac{1}{\lambda j+\alpha}\ {}_2F_1\bigl(2\alpha, \alpha+\lambda j; \alpha+\lambda j+1; -1\bigr)\,,
$$

and

$$
\begin{aligned}
I_2 &= \int_0^{\exp(-x)} \frac{z^{\lambda j+\alpha-1}}{(1+z)^{2\alpha}}\, dz \\
(2.6)\qquad &= \frac{\exp\bigl(-(\alpha+\lambda j)\, y\bigr)}{\alpha+\lambda j}\ {}_2F_1\bigl(2\alpha, \alpha+\lambda j; \alpha+\lambda j+1; -\exp(-x)\bigr)\,.
\end{aligned}
$$

Combining (2.5) and (2.6) and substituting into (2.4), the cdf $F(x)$ for $x > 0$ becomes

$$
\begin{aligned}
F(x) &= F(0) + \frac{2}{B(\alpha,\alpha)^2} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}(-1)^i}{i+\alpha} \\
&\quad \cdot \Biggl\{ \frac{1}{\lambda j+\alpha}\ {}_2F_1\bigl(2\alpha, \alpha+\lambda j; \alpha+\lambda j+1; -1\bigr) \\
&\qquad - \frac{\exp\bigl(-(\alpha+\lambda j)\, x\bigr)}{\alpha+\lambda j}\ {}_2F_1\bigl(2\alpha, \alpha+\lambda j, \alpha+\lambda j+1; -\exp(-x)\bigr) \Biggr\}\,.
\end{aligned}
$$

Repeating the above argument with $x = 0$ yields the form for $F(0)$ as

$$\begin{aligned} F(0) &= \int_{-\infty}^{0} f(t)\,dt \\ &= \frac{2}{B(\alpha,\alpha)^2} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}(-1)^i}{i+\alpha} \\ &\quad \cdot \int_{-\infty}^{0} \frac{\exp\Big(t\big(\lambda(i+\alpha+j)+\alpha\big)\Big)}{\big(1+\exp(t)\big)^{2\alpha}}\,dt \\ &= \frac{2}{B(\alpha,\alpha)^2} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}(-1)^i}{i+\alpha} \int_0^1 \frac{z^{\lambda(i+\alpha+j)+\alpha-1}}{(1+z)^{2\alpha}}\,dz \\ &= \frac{2}{B(\alpha,\alpha)^2} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}(-1)^i}{(i+\alpha)} \\ &\quad \cdot \frac{{}_2F_1\Big(2\alpha,\ \lambda(i+\alpha+j)+\alpha;\ \lambda(i+\alpha+j)+\alpha+1;\ -1\Big)}{\lambda(i+\alpha+j)+\alpha}\ . \end{aligned}$$

If $x < 0$, then similar arguments by using equation (3.194.1) in Gradshteyn and Ryzhik [8] yields

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \frac{2}{B(\alpha,\alpha)^2} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}(-1)^i}{i+\alpha}\, I\ ,$$

where

$$\begin{aligned} I &= \int_{-\infty}^{x} \frac{\exp\Big(t\big(\lambda(j+\alpha+i)+\alpha\big)\Big)}{\big(1+\exp(t)\big)^{2\alpha}} = \int_0^{\exp(x)} \frac{z^{\lambda(i+\alpha+j)+\alpha-1}}{(1+z)^{2\alpha}}\,dz \\ &= \frac{\exp\Big(x\big(\lambda(i+\alpha+j)+\alpha\big)\Big)}{\lambda(i+\alpha+j)+\alpha} \\ &\quad \cdot {}_2F_1\Big(2\alpha,\ \lambda(i+\alpha+j)+\alpha;\ \lambda(i+\alpha+j)+\alpha+1,\ -\exp(x)\Big)\,, \end{aligned}$$

and so the result

$$\begin{aligned} F(x) &= \frac{2}{B(\alpha,\alpha)^2} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}(-1)^i}{i+\alpha} \\ &\quad \cdot \frac{\exp\Big(x\big(\lambda(i+\alpha+j)+\alpha\big)\Big)}{\lambda(i+\alpha+j)+\alpha} \\ &\quad \cdot {}_2F_1\Big(2\alpha,\ \lambda(i+\alpha+j)+\alpha;\ \lambda(i+\alpha+j)+\alpha+1,\ -\exp(x)\Big)\,. \end{aligned}$$

Using the triple series representation, (1.5), the cdf $F(x)$ can be calculated as

$$F(x) = \begin{cases} 1 - \dfrac{2}{B(\alpha,\alpha)^2} \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \dfrac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}\binom{-2\alpha}{k}(-1)^i}{(i+\alpha)\,(\lambda j+k+\alpha)} \\ \quad \cdot\ \exp\bigl(-x(\lambda j+k+\alpha)\bigr)\,, & x>0\,, \\ \dfrac{2}{B(\alpha,\alpha)^2} \displaystyle\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \dfrac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}\binom{-2\alpha}{k}(-1)^i}{(i+\alpha)\,\bigl(\lambda(i+\alpha+j)+k+\alpha\bigr)} \\ \quad \cdot\ \exp\Bigl(x\bigl(\lambda(i+\alpha+j)+k+\alpha\bigr)\Bigr)\,, & x<0\,. \end{cases}$$

3. MOMENTS

Many of the interesting characteristics of the general skew logistic distribution can be studied through its moments. Let $X \sim \text{GSL}(\alpha,\lambda)$. In this section, we derive the nth moment of X. It is easy to show that if X follows $\text{GSL}(\alpha,\lambda)$ then $Y=|X|$ has the folded form of the type III generalized logistic distribution specified by the pdf

$$g(y;\alpha,\lambda) = \frac{2}{B(\alpha,\alpha)} \frac{\exp(-\alpha y)}{\bigl(1+\exp(-y)\bigr)^{2\alpha}}$$

for $y>0$. Thus, the even order moments of X are obtained as

$$\begin{aligned} E\left(X^n\right) &= \frac{2}{B(\alpha,\alpha)} \int_0^{\infty} \frac{x^n \exp(-\alpha x)}{\bigl(1+\exp(-x)\bigr)^{2\alpha}}\, dx = \frac{2}{B(\alpha,\alpha)} \int_0^1 \frac{\left(\ln\frac{1}{z}\right)^n z^{\alpha-1}}{(1+z)^{2\alpha}}\, dz \\ &= \frac{2}{B(\alpha,\alpha)} \sum_{i=0}^{\infty} \binom{-2\alpha}{i} \int_0^1 \left(\ln\frac{1}{z}\right)^n z^{\alpha+i-1}\, dz \\ &= \frac{2n!}{B(\alpha,\alpha)} \sum_{i=0}^{\infty} \frac{\binom{-2\alpha}{i}}{(\alpha+i)^{n+1}} = \frac{2n!}{B(\alpha,\alpha)}\, \tau_1(n+1,\alpha)\,, \end{aligned}$$

where the penultimate step follows by using equation (4.272.6) in Gradshteyn and Ryzhik [8].

If n is odd then, using the triple series representation, (1.5), one obtains

$$
\begin{aligned}
E\left(X^n\right) &= \frac{2}{B^2(\alpha,\alpha)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}\binom{-2\alpha}{k}(-1)^i}{i+\alpha} \\
&\quad \cdot \left\{ \int_{-\infty}^{0} x^n \exp\Big(x\big(\lambda(i+\alpha+j)+k+\alpha\big)\Big)\, dx \right. \\
&\quad \left. + \int_{0}^{\infty} x^n \exp\Big(-x\big(\lambda j+k+\alpha\big)\Big)\, dx \right\} \\
&= \frac{2n!}{B^2(\alpha,\alpha)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}\binom{-2\alpha}{k}(-1)^i}{i+\alpha} \\
&\quad \cdot \left\{ \frac{1}{(\lambda j+k+\alpha)^{n+1}} - \frac{1}{\big(\lambda(i+\alpha+j)+k+\alpha\big)^{n+1}} \right\} \\
&= \frac{2n!}{B^2(\alpha,\alpha)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}(-1)^i}{i+\alpha} \Delta(n+1,\alpha,\lambda)\ ,
\end{aligned}
$$

where $\Delta(n+1,\alpha,\lambda) = \tau_2(n+1,j,\alpha,\lambda) - \tau_2(n+1,i+\alpha+j,\alpha,\lambda)$.

Using these, the first four moments of X can be obtained as

$$
\begin{aligned}
E(X) &= \frac{2}{B^2(\alpha,\alpha)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}(-1)^i}{i+\alpha} \Delta(2,\alpha,\lambda)\ , \\
E(X^2) &= \frac{4}{B(\alpha,\alpha)} \tau_1(3,\alpha)\ , \\
E(X^3) &= \frac{12}{B^2(\alpha,\alpha)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\binom{\alpha-1}{i}\binom{-(\alpha+i)}{j}(-1)^i}{i+\alpha} \Delta(4,\alpha,\lambda)\ ,
\end{aligned}
$$

and

$$
E(X^4) = \frac{48}{B(\alpha,\alpha)} \tau_1(5,\alpha)\ .
$$

Using the above moments, we can calculate the four measures $E(X)$, $Var(X)$, $Skewness(X)$ and $Kurtosis(X)$. Figures 3 to 6 illustrate the behavior of the four measures for $\lambda = -10, ..., 10$ and $\alpha = 1, 2, 5$. From these figures, we see that:

- **(i)** $E(X)$ increases with increasing λ;
- **(ii)** $E(X)$ decreases with increasing α;
- **(iii)** $Var(X)$ decreases with increasing $|\lambda|$;

(iv) $Var(X)$ decreases with increasing α;

(v) $Skewness(X)$ increases with increasing λ;

(vi) $|Skewness(X)|$ decreases with increasing α;

(vii) $Kurtosis(X)$ initially decreases before increasing with increasing $|\lambda|$;

(viii) $Kurtosis(X)$ decreases with increasing α.

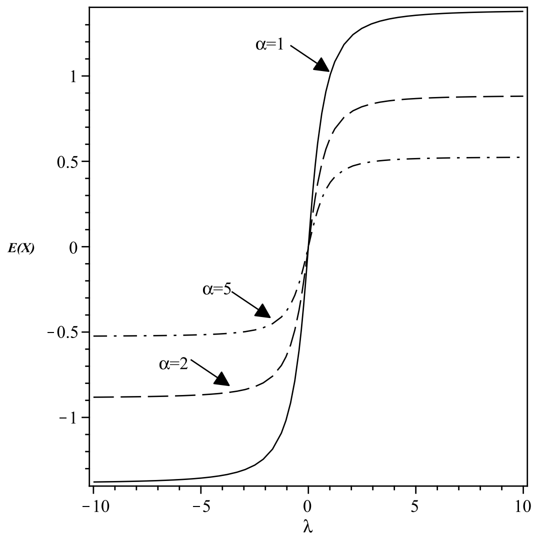

Figure 3: Plot of $E(X)$.

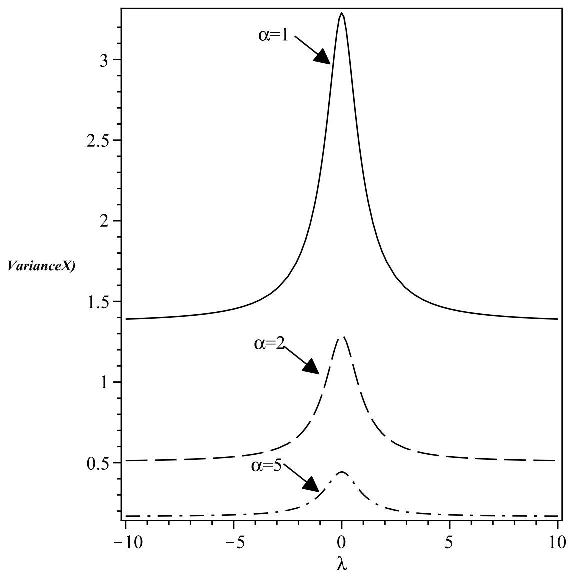

Figure 4: Plot of $Variance(X)$.

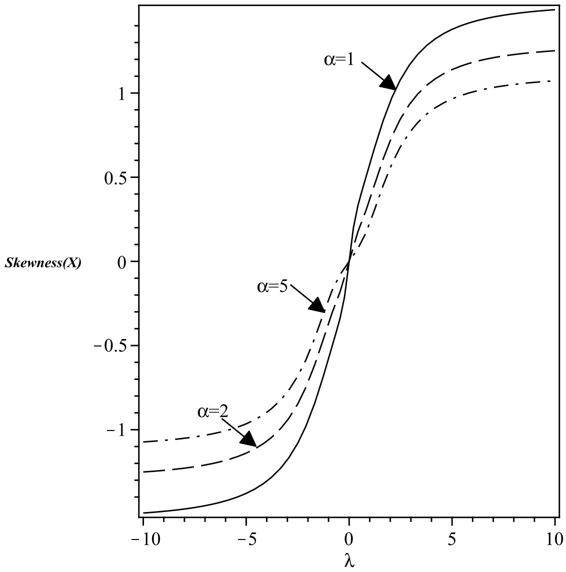

Figure 5: Plot of $Skewness(X)$.

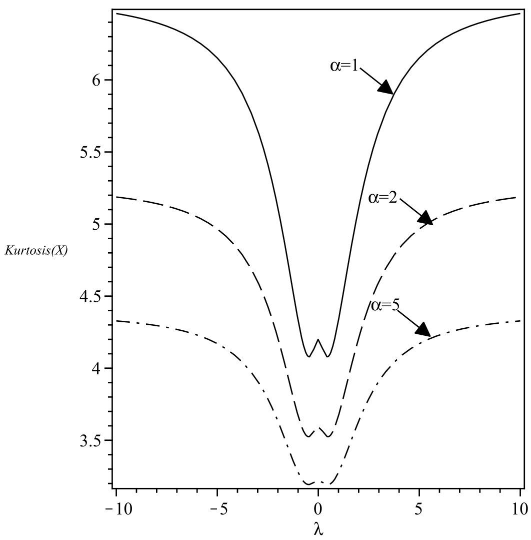

Figure 6: Plot of $Kurtosis(X)$.

4. ESTIMATION

Let us first consider a version of (1.3) with the location parameter $\mu \in R$ and scale parameter $\sigma > 0$, i.e.,

$$f(x;\mu,\sigma,\alpha,\lambda) = \frac{2}{\sigma}\, g_\alpha\left(\frac{x-\mu}{\sigma}\right) G_\alpha\left[\lambda\left(\frac{x-\mu}{\sigma}\right)\right] \tag{4.1}$$

for $-\infty < x < \infty$, $\alpha > 0$ and $\lambda \in R$. In this section, we consider estimation of the parameters μ, σ, α and λ and provide expressions for the Fisher information matrix. The log-likelihood for a random sample $x_1, ..., x_n$ from (4.1) is:

$$\ell = \ln L(\mu,\sigma,\alpha,\lambda) = -n\ln\sigma + \sum_{i=1}^{n} \ln g_\alpha(y_i) + \sum_{i=1}^{n} \zeta_0(\lambda y_i) \ , \tag{4.2}$$

where $y_i = \frac{x_i-\mu}{\sigma}$ and $\zeta_0(x) = \ln\{2G_\alpha(x)\}$. We also define the derivative $\zeta_m(x) = d^m\zeta_0(x)/dx^m$, $m = 1, 2, 3, ...$ and note that $\zeta_1(x) = g_\alpha(x)/G_\alpha(x)$. All subsequent derivatives can be expressed as functions of $\zeta_1(x)$; in particular, $\zeta_2(x) = -\alpha\left(\frac{1-\exp(-x)}{1+\exp(-x)}\right)\zeta_1(x) - \zeta_1(x)^2$.

By differentiating (4.2) with respect to μ, σ, α and λ, and equating the derivatives to zero, the maximum likelihood estimators are the simultaneous solutions of

$$2\alpha \sum_{i=1}^{n} \frac{\exp(-y_i)}{1+\exp(-y_i)} + \lambda \sum_{i=1}^{n} \zeta_1(\lambda y_i) = n\,\alpha \ , \tag{4.3}$$

$$n + \lambda \sum_{i=1}^{n} y_i\, \zeta_1(\lambda y_i) = \alpha \sum_{i=1}^{n} \frac{y_i\big(1-\exp(-y_i)\big)}{1+\exp(-y_i)} \ , \tag{4.4}$$

$$\sum_{i=1}^{n} y_i + 2\sum_{i=1}^{n} \ln\big\{1+\exp(-y_i)\big\} - \sum_{i=1}^{n} \frac{\partial \ln\big\{2G_\alpha(\lambda y_i)\big\}}{\partial\alpha} = 2n\big(\Psi(2\alpha) - \Psi(\alpha)\big) \tag{4.5}$$

and

$$\sum_{i=1}^{n} y_i\, \zeta_1(\lambda y_i) = 0 \ , \tag{4.6}$$

where $\Psi(x) = \ln\Gamma(x)/dx$ is the digamma function. In (4.5), we have

$$\frac{\partial \ln\big\{2G_\alpha(\lambda y)\big\}}{\partial\alpha} = 2\big(\Psi(2\alpha) - \Psi(\alpha)\big) - \frac{\displaystyle\int_{-\infty}^{\lambda y} t\, g_\alpha(t)\, dt}{G_\alpha(\lambda y)} - 2\, \frac{\displaystyle\int_{-\infty}^{\lambda y} \ln\big(1+\exp(-t)\big)\, g_\alpha(t)\, dt}{G_\alpha(\lambda y)} \ .$$

The maximum likelihood estimators $(\widehat{\mu}, \widehat{\sigma}, \widehat{\lambda}, \widehat{\alpha})$ of $(\mu, \sigma, \lambda, \alpha)$ are consistent estimators, and $\sqrt{n}(\widehat{\mu} - \mu, \widehat{\sigma} - \sigma, \widehat{\lambda} - \lambda, \widehat{\alpha} - \alpha)$ is asymptotically normal with zero means and variance covariance matrix $\mathbf{I}^{-1}$, where

$$\mathbf{I} = -\frac{1}{n} \begin{bmatrix} E\left(\frac{\partial^2 \ell}{\partial^2 \mu}\right) & E\left(\frac{\partial^2 \ell}{\partial \mu \partial \sigma}\right) & E\left(\frac{\partial^2 \ell}{\partial \mu \partial \lambda}\right) & E\left(\frac{\partial^2 \ell}{\partial \mu \partial \alpha}\right) \\ E\left(\frac{\partial^2 \ell}{\partial \sigma \partial \mu}\right) & E\left(\frac{\partial^2 \ell}{\partial^2 \sigma}\right) & E\left(\frac{\partial^2 \ell}{\partial \sigma \partial \lambda}\right) & E\left(\frac{\partial^2 \ell}{\partial \sigma \partial \alpha}\right) \\ E\left(\frac{\partial^2 \ell}{\partial \lambda \partial \mu}\right) & E\left(\frac{\partial^2 \ell}{\partial \lambda \partial \sigma}\right) & E\left(\frac{\partial^2 \ell}{\partial^2 \lambda}\right) & E\left(\frac{\partial^2 \ell}{\partial \lambda \partial \alpha}\right) \\ E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \mu}\right) & E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \sigma}\right) & E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \lambda}\right) & E\left(\frac{\partial^2 \ell}{\partial^2 \alpha}\right) \end{bmatrix}.$$

Now, we compute the Fisher information matrix based on the likelihood equations. These enable, for example, construction of confidence intervals based on pivotal quantities using the limiting normal distribution. For simplicity, let us consider interval estimation of (μ, σ, λ) when α is known. In this case, the elements of the Fisher information matrix can be written as

$$\begin{aligned}
-E\left(\frac{\partial^2 \ell}{\partial^2 \mu}\right) &= \frac{2\alpha n}{\sigma^2} I_1 + \frac{n\lambda^2}{\sigma^2} I_2 \ , \\
-E\left(\frac{\partial^2 \ell}{\partial^2 \sigma}\right) &= -\frac{n}{\sigma^2} + \frac{2n\alpha}{\sigma^2} E(Z) - \frac{4n\alpha}{\sigma^2} I_3 + \frac{2n\alpha}{\sigma^2} I_4 + \frac{n\lambda^2}{\sigma^2} a_{22}(\lambda) \ , \\
-E\left(\frac{\partial^2 \ell}{\partial^2 \lambda}\right) &= n a_{22}(\lambda) \ , \\
-E\left(\frac{\partial^2 \ell}{\partial \mu \partial \lambda}\right) &= \frac{n}{\sigma} I_5 - \frac{n\alpha\lambda}{\sigma} I_6 - \frac{n\lambda}{\sigma} a_{12}(\lambda) \ , \\
-E\left(\frac{\partial^2 \ell}{\partial \sigma \partial \lambda}\right) &= \frac{n\alpha\lambda}{\sigma} I_6 - \frac{n\lambda}{\sigma} a_{12}(\lambda) \ , \\
-E\left(\frac{\partial^2 \ell}{\partial \mu \partial \sigma}\right) &= \frac{n\alpha}{\sigma^2} + \frac{2n\alpha}{\sigma^2} I_1 - \frac{2n\alpha}{\sigma^2} I_7 - \frac{n\lambda}{\sigma^2} I_5 - \frac{n\lambda^2\alpha}{\sigma^2} I_6 - \frac{n\lambda^2}{\sigma^2} a_{12}(\lambda) \ , \\
-E\left(\frac{\partial^2 \ell}{\partial^2 \alpha}\right) &= 4n\big(\Psi(1,\alpha) - 2\Psi(1,2\alpha)\big) - nI_8 - 4nI_9 - 4nI_{10} \\
&\quad - nI_{11} + 4nI_{12} + 4nI_{13} \ , \\
-E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \mu}\right) &= -\frac{n}{\sigma} I_{17} - \frac{2n\lambda}{\sigma} I_{18} + \frac{n\lambda}{\sigma} I_{19} + \frac{n\lambda}{\sigma} I_{20} \ , \\
-E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \sigma}\right) &= \frac{n}{\sigma} I_{21} - \frac{2n\lambda}{\sigma}\big(\Psi(\alpha) - \Psi(2\alpha)\big) I_5 - \frac{n\lambda^2}{\sigma} a_{21}(\lambda) - \frac{2n\lambda}{\sigma} I_{23} \\
&\quad + \frac{n\lambda}{\sigma} I_{22} + \frac{2n\lambda}{\sigma} I_{16} \ , \\
-E\left(\frac{\partial^2 \ell}{\partial \alpha \partial \lambda}\right) &= 2nI_{14} - nI_{15} - nI_{16} \ ,
\end{aligned}$$

where

$$I_1 = E\left(\frac{\exp(-Z)}{1+\exp(-Z)}\right), \quad I_2 = E\left(\zeta_1^2(\lambda Z)\right), \quad I_3 = E\left(\frac{Z\exp(-Z)}{1+\exp(-Z)}\right),$$

$$I_4 = E\left(\frac{Z^2\exp(-Z)}{\left\{1+\exp(-Z)\right\}^2}\right), \quad I_5 = E\left(\zeta_1(\lambda Z)\right),$$

$$I_6 = E\left(\frac{Z\left(1-\exp(-\lambda Z)\right)\zeta_1(\lambda Z)}{1+\exp(-\lambda Z)}\right), \quad I_7 = E\left(\frac{Z\exp(-Z)}{\left\{1+\exp(-Z)\right\}^2}\right),$$

$$I_8 = E\left(\frac{b_2(\lambda Z)}{G_\alpha(\lambda Z)}\right), \quad I_9 = E\left(\frac{c_{11}(\lambda Z)}{G_\alpha(\lambda Z)}\right), \quad I_{10} = E\left(\frac{c_{02}(\lambda Z)}{G_\alpha(\lambda Z)}\right),$$

$$I_{11} = E\left(\frac{b_1^2(\lambda Z)}{G_\alpha^2(\lambda Z)}\right), \quad I_{12} = E\left(\frac{c_{01}^2(\lambda Z)}{G_\alpha^2(\lambda Z)}\right), \quad I_{13} = E\left(\frac{b_1(\lambda Z)\,c_{01}(\lambda Z)}{G_\alpha^2(\lambda Z)}\right),$$

$$I_{14} = E\Big(Z\,\zeta_1(\lambda Z)\,\ln\big(1+\exp(-\lambda Z)\big)\Big), \quad I_{15} = E\left(Z\,\zeta_1(\lambda Z)\,\frac{b_1(\lambda Z)}{G_\alpha(\lambda Z)}\right),$$

$$I_{16} = E\left(Z\,\zeta_1(\lambda Z)\,\frac{c_{01}(\lambda Z)}{G_\alpha(\lambda Z)}\right), \quad I_{17} = E\left(\frac{1-\exp(-Z)}{1+\exp(-Z)}\right),$$

$$I_{18} = E\Big(\zeta_1(\lambda Z)\,\ln\big(1+\exp(-\lambda Z)\big)\Big), \quad I_{19} = E\left(\zeta_1(\lambda Z)\,\frac{b_1(\lambda Z)}{G_\alpha(\lambda Z)}\right),$$

$$I_{20} = E\left(\zeta_1(\lambda Z)\,\frac{c_{01}(\lambda Z)}{G_\alpha(\lambda Z)}\right), \quad I_{21} = E\left(Z\,\frac{\exp(-Z)-1}{\exp(-Z)+1}\right),$$

$$I_{22} = E\Big(Z\,\zeta_1(\lambda Z)\,b_1(\lambda Z)\Big), \quad I_{23} = E\Big(Z\,\zeta_1(\lambda Z)\,\ln\big(1+\exp(-Z)\big)\Big),$$

where $Z = (X-\mu)/\sigma$,

$$a_{kh}(\lambda) = E_\lambda\left\{Z^k\,\zeta_1^h(\lambda Z)\right\}, \quad b_k(x) = \int_{-\infty}^{x} t^k\,g_\alpha(t)\,dt\,, \quad \Psi(n,x) = \frac{d^n}{d^n x}\Psi(x)$$

and

$$c_{kh}(x) = \int_{-\infty}^{x} t^k\,\ln^h\left\{1+\exp(-t)\right\}g_\alpha(t)\,dt\;.$$

Note that $a_{k1}(\lambda)=0$ when k is odd, and that $a_{kh}(\lambda)\geq 0$ when both k and h are even. Also $E(h(Z)\,\zeta_1(\lambda Z))=0$ when $h(x)$ is an odd function and $E(h(Z)\,\zeta_1(\lambda Z))\geq 0$ when $h(x)$ is a even function. In general, these expectations will have to be computed numerically. However, closed-form expressions are possible in some particular cases.

5. REAL DATA APPLICATIONS

In this section, we fit the general skew logistic GSL$(\mu, \sigma, \lambda, \alpha)$ distribution to two real data sets. We compare the fits with those of the usual logistic distribution L(μ, σ), the type III generalized logistic distribution GL(μ, σ, α), the skew logistic distribution SL(μ, σ, λ), Azzalini's [2] skew normal distribution SN(μ, σ, λ), and Azzalini and Capitanio's [5] skew t distribution ST$(\mu, \sigma, \lambda,\ \alpha)$. The parameter λ in the skew normal and skew t distributions is the skewness parameter. The parameter α in the skew t distribution is the degree of freedom parameter. As with Azzalini's [2] skew normal distribution, Azzalini and Capitanio's [5] skew t distribution has been studied by many authors. Two most recent papers are Arellano-Valle and Azzalini [1] and Azzalini and Arellano-Valle [4].

Example 1. The first data set represents the strength data originally reported in Badar and Priest [6]. It represents the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge length of 10mm. This data have been analyzed previously by Raqab and Kundu [13] and Gupta and Kundu [9]. The data are as follows:

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397
2.445	2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614
2.616	2.618	2.624	2.659	2.675	2.738	2.740	2.856	2.917
2.928	2.937	2.937	2.977	2.996	3.030	3.125	3.139	3.145
3.220	3.223	3.235	3.243	3.264	3.272	3.294	3.332	3.346
3.377	3.408	3.435	3.493	3.501	3.537	3.554	3.562	3.628
3.852	3.871	3.886	3.971	4.024	4.027	4.225	4.395	5.020.

We fitted all six distributions to the above data by the method of maximum likelihood. The GSL distribution was fitted by solving (4.3)–(4.6). Table 1 presents the parameter estimates, the log likelihoods (LL), the Kolmogorov–Smirnov (K-S) statistics and respective p-values. Table 2 presents the chi-squared statistics with observed and expected frequencies. Note that the last two columns of Tables 1 and 2 appear identical. This can be explained by the well-known fact that the ST distribution reduces to the SN distribution as α approaches infinity.

Note also that the $\widehat{\alpha}$ for the GSL distribution is very large. Some elementary calculations show that

$$g_\lambda(x) \;\to\; \frac{1}{4^\alpha B(\alpha,\alpha)}\, I\{x=0\}$$

and

$$G_\lambda(x) \;\to\; I\{x \ge 0\}$$

as $\alpha \to \infty$. So, the pdf of the GSL distribution in (1.3) reduces to

$$f(x;\alpha,\lambda) \;\to\; \frac{2^{1-2\alpha}}{B(\alpha,\alpha)}\, I\{x=0\}\, I\{\lambda x \geq 0\}$$

as $\alpha \to \infty$.

Table 1: MLEs, log-likelihoods, Kolmogorov–Smirnov statistics and corresponding p-values for Example 1.

Distribution	L(μ,σ)	SL(μ,σ,λ)	GL(μ,σ,α)	GSL$(\mu,\sigma,\lambda,\alpha)$	SN(μ,σ,λ)	ST$(\mu,\sigma,\lambda,\alpha)$
$\widehat{\mu}$	3.024	2.328	3.048	2.271	2.271	2.271
$\widehat{\sigma}$	0.352	0.550	0.930	195.801	1.000	1.000
$\widehat{\lambda}$	—	3.713	—	4.418	4.419	4.419
$\widehat{\alpha}$	—	—	5.041879	76605.63	—	26491.46
Log-likelihood	−59.330	−56.794	−58.797	−55.902	−55.902	−55.902
KSS	0.094	0.084	0.097	0.073	0.075	0.075
p-value	0.606	0.754	0.571	0.880	0.877	0.877

Table 2: Observed and expected frequencies and chi-squared statistics for Example 1.

Intervals	Observed	L(μ,σ)	SL(μ,σ,λ)	GL(μ,σ,α)	GSL$(\mu,\sigma,\lambda,\alpha)$	SN(μ,σ,λ)	ST$(\mu,\sigma,\lambda,\alpha)$
< 2.5	12	11.61	11.88	11.48	12.28	12.28	12.28
2.5–3.0	20	18.80	22.53	18.00	21.35	21.35	21.35
3.0–3.5	17	19.61	15.25	19.22	15.55	15.55	15.55
3.5–4	9	9.26	7.61	10.51	8.53	8.53	8.53
> 4	5	3.72	5.74	3.79	5.29	5.29	5.29
		$\chi^2=0.8832$	$\chi^2=0.8345$	$\chi^2=1.1055$	$\chi^2=0.2682$	$\chi^2=0.2684$	$\chi^2=0.2684$

From Tables 1 and 2, we see that the GSL distribution provides a better fit for the data than the other five distributions. The GSL distribution takes the smallest chi-squared statistic, the smallest K-S statistic, and the largest p-value. The SN and ST distributions take the second smallest chi-squared statistic, the second smallest K-S statistic, and the second largest p-value. The largest log-likelihood of -55.902 is shared by the GSL, SN and ST distributions. Because of this, one can argue that the SN distribution is a competitor to the GSL distribution (or perhaps that the SN distribution is a better choice than the GSL distribution) since the former has one less parameter.

Figure 7 plots the fitted pdfs on top of the empirical histogram of the data. Figure 8 plots the fitted cdfs on top of the empirical cdf of the data. Both these figures support conclusions based on Tables 1 and 2. In both these figures, the fitted pdfs for the GSL, SN and ST distributions appear almost indistinguishable. Both figures suggest that the GSL distribution captures the tails of the data better than most other distributions.

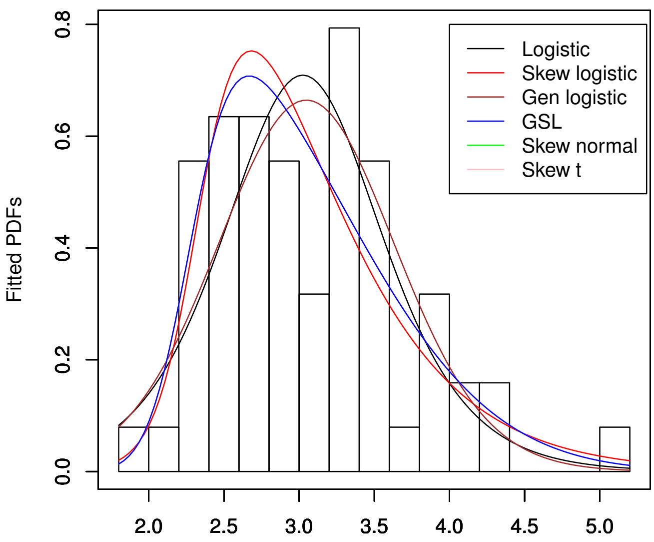

Figure 7: Histogram of the first data set and the fitted pdfs.

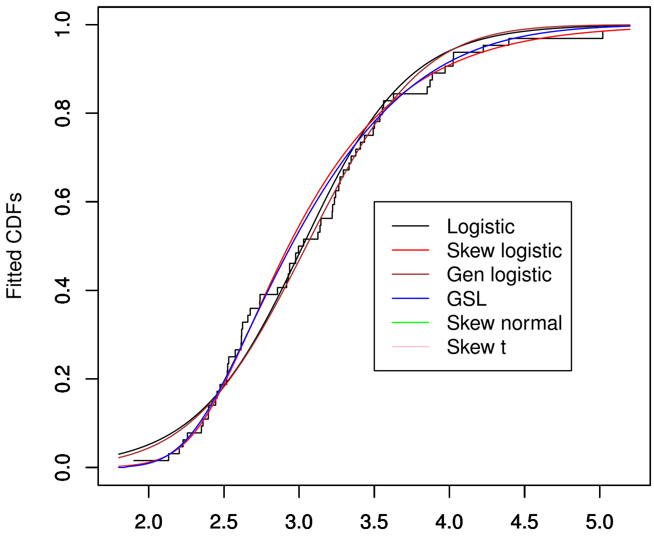

Figure 8: Empirical cdf of the first data set and the fitted cdfs.

Example 2. Here, we analyze the lean body mass of Australian athletes. The data given in Cook and Weisberg [7] are as follows:

63.32	58.55	55.36	57.18	53.2	53.77	60.17	48.33	54.57	53.42
68.53	61.85	48.32	66.24	57.92	56.52	54.78	56.31	62.96	56.68
62.39	63.05	56.05	53.65	65.45	64.62	60.05	56.48	41.54	52.78
52.72	61.29	59.59	61.7	62.46	53.14	47.09	53.44	48.78	56.05
56.45	53.11	54.41	55.97	51.62	58.27	57.28	57.3	54.18	42.96
54.46	57.2	54.38	57.58	61.46	53.46	54.11	55.35	55.39	52.23
59.33	61.63	63.39	60.22	55.73	48.57	51.99	51.17	57.54	68.86
63.04	63.03	66.85	59.89	72.98	45.23	55.06	46.96	53.54	47.57
54.63	46.31	49.13	53.71	53.11	46.12	53.41	51.48	53.2	56.58
56.01	46.52	51.75	42.15	48.76	41.93	42.95	38.3	34.36	39.03

We fitted all six distributions to the above data by the method of maximum likelihood. Table 3 presents the parameter estimates, the log likelihoods, the Kolmogorov–Smirnov statistics and respective p-values. The corresponding chi-squared statistics with observed and expected frequencies are reported in Table 4.

Table 3: MLEs, log-likelihoods, Kolmogorov–Smirnov statistics and corresponding p-values for Example 2.

Distribution	L(μ,σ)	SL(μ,σ,λ)	GL(μ,σ,α)	GSL$(\mu,\sigma,\lambda,\alpha)$	SN(μ,σ,λ)	ST$(\mu,\sigma,\lambda,\alpha)$
$\widehat{\mu}$	55.101	57.148	55.036	55.356	54.895	59.085
$\widehat{\sigma}$	3.807	3.990	0.593	0.671	6.887	7.233
$\widehat{\lambda}$	—	-0.389	—	-0.057	8.706×10^{-6}	-0.903
$\widehat{\alpha}$	—	—	0.117	0.133	—	9.924
Log-likelihood	-334.013	-333.557	-333.333	-333.265	-334.865	-333.738
KSS	0.072	0.071	0.070	0.069	0.080	0.072
p-value	0.658	0.712	0.712	0.715	0.642	0.711

Table 4: Observed and expected frequencies and chi-squared statistics for Example 2.

Intervals	Observed	L(μ,σ)	SL(μ,σ,λ)	GL(μ,σ,α)	GSL$(\mu,\sigma,\lambda,\alpha)$	SN(μ,σ,λ)	ST$(\mu,\sigma,\lambda,\alpha)$
< 42.084	5	3.17	3.85	3.97	4.35	3.14	3.93
42.084–49.808	16	16.77	16.69	14.22	14.47	19.86	16.95
49.808–57.532	48	45.51	44.28	50.67	50.18	41.90	43.57
57.532–65.256	25	28.06	29.66	24.35	24.78	28.47	30.15
> 65.256	6	6.49	5.52	6.79	6.23	6.62	5.40
		$\chi^2=1.5987$	$\chi^2=1.4574$	$\chi^2=0.7444$	$\chi^2=0.3633$	$\chi^2=3.2157$	$\chi^2=1.7422$

From Tables 3 and 4, we can see that the GSL distribution takes the largest log likelihood, the smallest chi-squared statistic, the smallest K-S statistic, and the largest p-value. The GL distribution takes the second largest log likelihood, the second smallest chi-squared statistic, the second smallest K-S statistic, and the second largest p-value. The SN distribution takes the smallest log likelihood, the largest chi-squared statistic, the largest K-S statistic, and the smallest p-value.

Figures 9 and 10 plot the fitted pdfs and fitted cdfs, respectively. Both these figures support conclusions based on Tables 3 and 4. Both figures suggest that the GSL distribution captures the middle part of the data better than most other distributions.

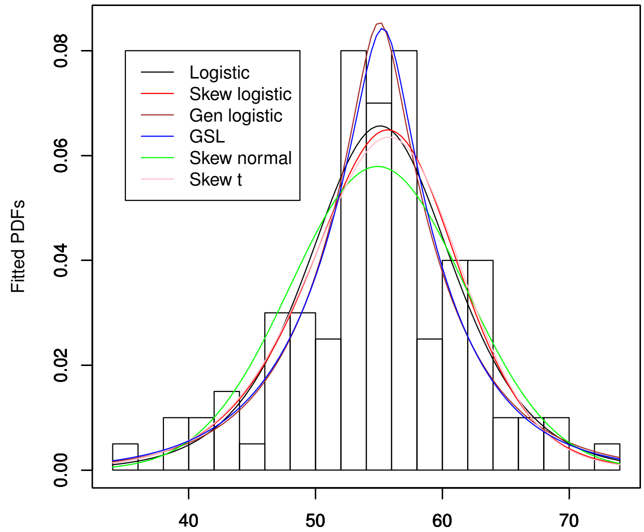

Figure 9: Histogram of the second data set and the fitted pdfs.

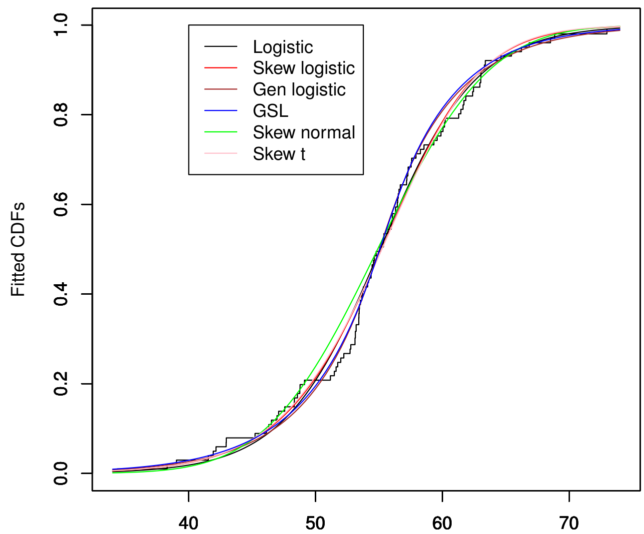

Figure 10: Empirical cdf of the second data set and the fitted cdfs.

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