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INFERENCES FOR THE CHANGE-POINT OF THE EXPONENTIATED WEIBULL HAZARD FUNCTION

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Abstract:

- In many applications of lifetime data analysis, it is important to perform inferences about the change-point of the hazard function. The change-point could be a maximum for unimodal hazard functions or a minimum for bathtub forms of hazard functions and is usually of great interest in medical or industrial applications. For lifetime distributions where this change-point of the hazard function can be analytically calculated, its maximum likelihood estimator is easily obtained from the invariance properties of the maximum likelihood estimators. From the asymptotical normality of the maximum likelihood estimators, confidence intervals can also be obtained. Considering the exponentiated Weibull distribution for the lifetime data, we have different forms for the hazard function: constant, increasing, unimodal, decreasing or bathtub forms. This model gives great flexibility of fit, but we do not have analytic expressions for the change-point of the hazard function. In this way, we consider the use of Markov Chain Monte Carlo methods to get posterior summaries for the change-point of the hazard function considering the exponentiated Weibull distribution.

Key-Words:

- *change-point; exponentiated Weibull distribution; hazard function; lifetime data analysis; Markov Chain Monte Carlo.*

AMS Subject Classification:

- 49A05, 78B26.

1. INTRODUCTION

Hazard function plays an important role in reliability and survival analysis and usually it can increase(decrease) up to a maximum(minimum) and then decrease(increase) after this change-point, also known as turning or critical point, [1, 12]. This is common in medical studies as in heart or kidney transplantation, where the patients have an increasing hazard during an adaptation period and a decreasing hazard after this adaptation period, [4]. In a study of recovery from breast cancer, it has been observed by [18] that the maximum mortality occurs after about three years and then it decreases slowly over a fixed period of time. In other situation, we have a bathtub shape for the hazard function where we have a decreasing hazard down to a minimum and then an increase after this change-point. In reliability, the turning point of a hazard function is useful in assessing the hazard in the useful life phase and this helps to determine and plan appropriate strategies for burn-in, maintenance and repair policies, [1].

Some common lifetime distributions like the exponential or Weibull are not appropriate to model non-monotonic hazard rate. Many existing probability distributions used to analyze lifetime data have unimodal hazard functions: the log-logistic distribution, [2]; the log-normal distribution, [17]; the inverse-Weibull distribution, [16]; the exponentiated Weibull distribution [20, 21] among many others. The exponentiated Weibull distribution is very flexible to be fitted by the data since it has constant, increasing, decreasing, unimodal and bathtub hazard functions. For situations where the hazard function is unimodal (bathtub) shaped, usually, we have interest in the estimation of the lifetime change-point that is, the point at which the hazard function reaches its maximum (minimum). In applications, the exponentiated Weibull distribution gives great flexibility of fit, but we do not have analytic expressions for the change-point of the hazard function.

In this paper, under the Bayesian point of view, we consider the use of Markov Chain Monte Carlo methods to get posterior summaries for the change-point of the hazard function. The maximum likelihood estimation procedure is also considered. It is important to point out that we do not have analytical expressions for this hazard change-point so we can not obtain classical asymptotic confidence intervals for the hazard change-point. The paper is organized as follows: in Section 2 we introduce some characteristics of the exponentiated Weibull distribution; in Section 3 we introduce the likelihood function in the presence of censored observations; in Section 4 we introduce some illustrative examples and finally, in Section 5 we present some conclusions.

2. THE EXPONENTIATED WEIBULL DISTRIBUTION

Let T be a non-negative random variable with an exponentiated Weibull distribution with hazard function given by:

$$h(t) \,=\, \theta\, \frac{h_1(t)\, S_1(t)\, F_1(t)^{\theta-1}}{1 - \big[F_1(t)\big]^{\theta}} \tag{2.1}$$

where $h_1(t)$, $S_1(t)$ and $F_1(t)$ are, respectively, the hazard function, the survival function and the accumulated distribution function of the Weibull distribution, [19]. For the Weibull distribution with scale parameter $\mu > 0$ and shape parameter $\beta > 0$, we have:

$$h_1(t) = \frac{\beta}{\mu^\beta}\, t^{\beta-1}\,, \qquad S_1(t) = \exp\left[-\left(\frac{t}{\mu}\right)^{\beta}\right] \quad \text{and} \quad F_1(t) = 1 - \exp\left[-\left(\frac{t}{\mu}\right)^{\beta}\right]. \tag{2.2}$$

From the standard relations, [19], $S(t) = \exp\big[-\int_0^t h(u)\, du\big]$ and $f(t) = -\frac{d}{dt} S(t)$ and using the hazard function given in (2.1), we have the survival and density functions written as:

$$S(t) \,=\, 1 - \big[F_1(t)\big]^{\theta} \qquad \text{and} \qquad f(t) \,=\, \theta\, h_1(t)\, S_1(t)\, F_1(t)^{\theta-1} \tag{2.3}$$

respectively.

Explicitly we have that:

$$S(t) \,=\, 1 - \left\{1 - \exp\left[-\left(\frac{t}{\mu}\right)^{\beta}\right]\right\}^{\theta} \tag{2.4}$$

and

$$f(t) \,=\, \theta\, \frac{\beta}{\mu^\beta}\, t^{\beta-1} \exp\left[-\left(\frac{t}{\mu}\right)^{\beta}\right] \left\{1 - \exp\left[-\left(\frac{t}{\mu}\right)^{\beta}\right]\right\}^{\theta-1} \tag{2.5}$$

where $\mu > 0$ is the scale parameter and $\theta > 0$ and $\beta > 0$ are the shape parameters. For $\theta = 1$ in (2.4) and (2.5) we have the survival and density function for the two parameter Weibull distribution. While, taking $\beta = 1$ we have the exponentiated exponential distribution, introduced by [11].

The great advantage of the exponentiated Weibull distribution in comparison to Weibull distribution is related to the behavior of the hazard function which depends on the values of θ and β. In Figure 1, we have the different shapes of the hazard function given by (2.1).

β	θ	Hazard
$= 1$	$= 1$	Constant (I)
> 1	> 1	Increasing (II)
< 1	> 1	Unimodal (III)
< 1	< 1	Decreasing (IV)
> 1	< 1	Bathtub (V)

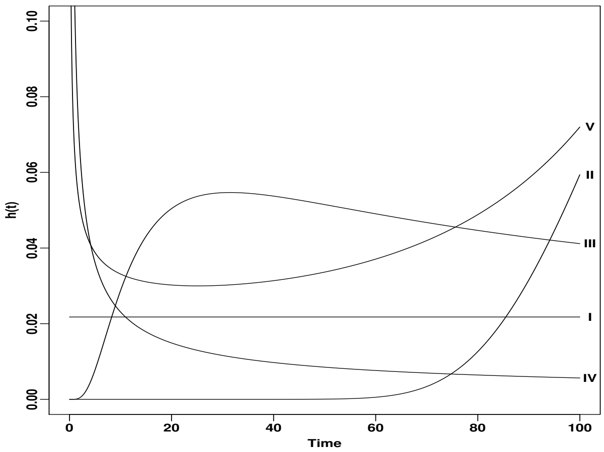

Figure 1: The exponentiated Weibull hazard function behavior.

3. HAZARD CHANGE POINT ESTIMATION — CLASSICAL APPROACH

Let us assume that $(t_1, \delta_1), ..., (t_n, \delta_n)$ is a random sample of size n of lifetimes generated by an exponentiated Weibull distribution with parameters μ, β and θ and that $(\delta_i = 1)$ if t_i is completely observed or $(\delta_i = 0)$ if t_i is a right censored observation $(i = 1, ..., n)$. Assuming a non-informative censoring mechanism, [19], the likelihood and log-likelihood functions are given, respectively, by:

$$L(\mu, \beta, \theta \mid \mathbf{t}, \boldsymbol{\delta}) = \prod_{i=1}^{n} \left(\theta \frac{h_1(t_i)\, S_1(t_i)\, F_1(t_i)^{\theta-1}}{1 - F_1(t_i)^{\theta}} \right)^{\delta_i} \left(1 - F_1(t_i)^{\theta}\right) \tag{3.1}$$

and:

$$\begin{aligned} l(\mu, \beta, \theta \mid \mathbf{t}, \boldsymbol{\delta}) &= R \log\theta + \sum_{i=1}^{n} \delta_i \log\bigl[f_1(t_i)\bigr] + (\theta - 1) \sum_{i=1}^{n} \delta_i \log\bigl[F_1(t_i)\bigr] \\ &\quad + \sum_{i=1}^{n} (1 - \delta_i)\, \log\bigl[1 - F_1(t_i)^{\theta}\bigr] \end{aligned} \tag{3.2}$$

where $R = \sum_{i=1}^{n} \delta_i$, $h_1(t_i)$, $S_1(t_i)$ and $F_1(t_i)$ are defined in (2.2) and $f_1(t_i) = h_1(t_i)\, S_1(t_i)$.

Given a vector of observed lifetimes $(t_1, t_2, ..., t_n)$ and defining $l = l(\mu, \beta, \theta \mid \mathbf{t}, \boldsymbol{\delta})$, the maximum likelihood estimates for μ, β and θ, denoted by $\hat{\mu}$, $\hat{\beta}$ and $\hat{\theta}$, are obtained solving, for example by Newton–Raphson, the following likelihood equations:

$$\frac{\partial}{\partial\mu} l \;=\; -\beta R + \sum_{i=1}^{n} \delta_i\, t_i\, h_1(t_i) - (\theta-1) \sum_{i=1}^{n} \delta_i\, t_i\, \frac{f_1(t_i)}{F_1(t_i)} + \sum_{i=1}^{n} (1-\delta_i)\, t_i\, h(t_i) \;=\; 0 \;,$$

$$\begin{aligned}\frac{\partial}{\partial\beta} l \;=\; & \sum_{i=1}^{n} (1-\delta_i)\, t_i\, h(t_i) \log\left(\frac{t_i}{\mu}\right) + (\theta-1) \sum_{i=1}^{n} \delta_i\, t_i\, \frac{f_1(t_i)}{F_1(t_i)} \log\left(\frac{t_i}{\mu}\right) \\ & - \sum_{i=1}^{n} t_i\, \delta_i\, h_1(t_i) \log\left(\frac{t_i}{\mu}\right) + R\big[(1-\beta)\log(\mu)\big] + \beta \sum_{i=1}^{n} \delta_i \log(t_i) \;=\; 0 \;,\end{aligned}$$

$$\frac{\partial}{\partial\theta} l \;=\; R + \theta \sum_{i=1}^{n} \delta_i \log F_1(t_i) + \theta \sum_{i=1}^{n} \frac{(\delta_i - 1)}{S(t)}\, F_1(t_i)^{\theta} \log F_1(t_i) \;=\; 0 \;.$$

The $100 \times (1-\alpha)\%$ confidence intervals for μ, β and θ can be obtained from the usual asymptotic normality of the maximum likelihood estimators with $\mathrm{Var}(\hat{\mu})$, $\mathrm{Var}(\hat{\beta})$ and $\mathrm{Var}(\hat{\theta})$ estimated from the inverse of the observed Fisher information matrix, that is, the inverse of the matrix of second derivatives of the log-likelihood function locally at $\hat{\mu}$, $\hat{\beta}$ and $\hat{\theta}$. From the invariance property of maximum likelihood estimators, we can obtain confidence intervals for functions of μ, β and θ. For $\phi = g(\mu,\beta,\theta)$, a differentiable function of μ, β and θ, we have $\hat{\phi} = g(\hat{\mu},\hat{\beta},\hat{\theta})$ and the variance of $\hat{\phi}$ is obtained using the delta method, [23].

Although the delta method is applied to estimate $\mathrm{Var}\big[g(\hat{\mu},\hat{\beta},\hat{\theta})\big]$, in some cases, it does not work, [5]. As a special situation, for $\beta < 1$ and $\theta > 1$ let us assume that we are interested in getting confidence intervals for the maximum of the exponentiated Weibull hazard function. Taking $\varphi = h(t)$, defined in (2.1), the maximum of the exponentiated Weibull hazard function, $T_{\max}$, is obtained as solution of the equation $\frac{d}{dt}\log(\varphi) = 0$ where, from (2.1):

$$\log(\varphi) \;\propto\; (\beta-1)\log(t) - \frac{t}{\beta}\, h_1(t) + (\theta-1)\log\big[F_1(t)\big] - \log\big[1 - F_1(t)^{\theta}\big] \tag{3.3}$$

and:

$$\frac{d}{dt}\log(\varphi) \;=\; \frac{\beta-1}{t} - h_1(t) + (\theta-1)\,\frac{h_1(t)\, S_1(t)}{F_1(t)} + \theta\,\frac{h_1(t)\, S_1(t)\, F_1(t)^{\theta-1}}{\big[1 - F_1(t)^{\theta}\big]} \;. \tag{3.4}$$

By the invariance principle of maximum likelihood estimator, the maximum likelihood estimator of the change point is the solution of (3.4) with μ, β and θ replaced by their maximum likelihood estimates. We observe that (3.4) is non-linear in t, so the maximum of the hazard function estimate $\hat{T}_{\max}$, should be obtained using some one dimensional root finding technique like Newton–Raphson. Since $\hat{T}_{\max}$ in not obtained from an analytical expression, it is not possible to estimate

$\mathrm{Var}\big[\hat{T}_{\max}\big]$ using the delta method. This fact shows the difficulty in applying the maximum likelihood methodology when the change point does not have closed form and this fact justify the application of the Bayesian methodology. Standard resampling procedures like the Bootstrap and the Jackknife are other alternatives but they will not be considered in this paper.

4. HAZARD CHANGE POINT ESTIMATION — BAYESIAN APPROACH

Under the Bayesian approach, assuming a joint prior distribution for $\boldsymbol{\nu} = (\mu,\beta,\theta)$ in the form $\pi(\boldsymbol{\nu}) = \pi(\theta)\,\pi(\beta)\,\pi(\theta)$, we get the joint posterior distribution given by:

$$\pi(\boldsymbol{\nu} \mid \mathbf{t},\boldsymbol{\delta}) \;\propto\; \pi(\boldsymbol{\nu}) \prod_{i=1}^{n} \left(\theta\,\frac{h_1(t_i)\,S_1(t_i)\,F_1(t_i)^{\theta-1}}{1-F_1(t_i)^{\theta}}\right)^{\delta_i} \big(1-F_1(t_i)^{\theta}\big) \;. \tag{4.1}$$

From (4.1) it is clear that is not possible to get explicit forms for the marginal posterior distributions for each parameter. In this way, we should use some approximation method to solve integrals as the Laplace method, [26], or some other numerical method, [22]. When models become too difficult to be analyze analytically, we have to use simulation algorithms, such as the Markov Chain Monte Carlo methods to obtain posterior estimates, [7, 14]. The Markov Chain Monte Carlo methods is a general simulation method for sampling from posterior distributions and computing posterior quantities of interest. To simulate samples of the joint posterior distribution of interest, we need to sample successively from a target distribution. The Gibbs algorithm requires to decompose the joint posterior distribution into full conditional distributions for each parameter in the model and then sample from each one of these conditional distributions

For the exponentiated Weibull distribution, the conditional posterior densities for μ, β and θ show that standard sampling schemes are not feasible since the conditional distributions are not given in a known form. In this way, an alternative target distribution to the full conditional distributions should be used. The alternative proposal distribution should be a distribution from which it is easy to sample from it; in this way, we use Metropolis–Hastings algorithms, [3, 14]. Tierney (1994) suggested, when possible, use of the Metropolis–Hastings algorithm within Gibbs sampling to sample from full conditional distributions.

In our applications, to sample from the full conditional distributions for μ, β and θ, we have used the Adaptive Rejection Metropolis Sampling algorithm, *ARMS*, introduced by Gilks *et al.* (1995) also discussed in [10]. This algorithm is a generalization of the method of adaptive rejection sampling of Gilks (1992) which includes a Metropolis step to accommodate non-log-concavity in the density that will be sampled. The *C* code, written by Gilks, and linked to the matrix language *Ox*, [6], was used to compute the posterior summaries of interest. The *ARMS* algorithm, to the best of our knowledge, also is available in the libraries *dlm*, *SamplerCompare* and *HI*, under *R,* [15]. *ARMS* is also available in *GENMOD*, *LIFEREG*, *PHREG* and *MCMC* procedures under *SAS* version 9.2. The MCMC procedure provides a flexible environment for fitting a wide range of models.

It is important to point out that [24] employed Bayesian and frequentist perspectives for estimating parameters for exponentiated Weibull distribution and showed a comprehensive and updating list of references.

5. SOME ILLUSTRATIVE EXAMPLES

5.1. An Example with a Simulated Data Set

In this subsection we introduce an example considering a simulated data set from an exponentiated Weibull distribution with parameters $\mu = 1$, $\beta = 0.8$ and $\theta = 4.0$. Since μ does not change the maximum of the hazard function, without loss of generality, we consider it known and equal to 1. Under this parameter configuration, the hazard function is of bathtub shape and we would like to estimate the parameter $T_{\max}$ (the "true value" of $T_{\max}$ is 3.9114). A total of $n = 50$ observations (see Table 1) were simulated using the inversion method considering $\delta_i = 1$ $(i = 1, ..., 50)$. Replacing the values of β and θ by their maximum likelihood or Bayesian estimates we can solve (3.4) for t to get the maximum of the hazard function.

Table 1: Simulated data set from an exponentiated Weibull distribution with $\mu = 1$, $\beta = 0.8$ and $\theta = 4.0$.

0.23	0.53	0.54	0.60	0.65	0.84	0.90	0.96	0.98	0.99
1.05	1.26	1.28	1.31	1.33	1.37	1.45	1.53	1.69	1.72
1.77	1.80	1.96	2.06	2.14	2.24	2.35	2.40	2.47	2.47
2.58	2.68	2.73	2.77	2.78	2.96	3.04	3.31	3.36	3.67
4.01	4.16	4.19	4.36	4.91	5.10	5.81	6.27	7.39	7.41

As observed in [1], the turning point of a hazard rate function is useful in assessing the hazard in the useful life phase and helps to determine and plan appropriate burn-in, maintenance, and repair policies and strategies. For many bathtub-shaped distributions, the turning point is unique, and the hazard varies little in the useful life phase.

In Table 2 we have the maximum likelihood estimates (standard-errors) and the posterior means of the parameters β, θ and $T_{\max}$ assuming non-informative gamma prior distribution for the parameters β and θ. The maximum likelihood estimates were obtained by the Newton–Raphson method available in the software *SAS/NLP* procedure, [13]. With the obtained maximum likelihood estimates $\hat{\beta}$ and $\hat{\theta}$ we estimate $T_{\max}$ maximizing (3.3). Under the Bayesian approach, the parameters β and θ were estimated using the *ARMS* algorithm in *Ox*. We simulated five separate chains using different overdispersed starting values for each run. The algorithm was run for 21 000 iterations and the starting values were based in previous runs of the *ARMS* algorithm for large intervals. This strategy follows the ideas discussed in Gilks *et al.* (1995, 1997). We considered five initial abscissae based on 5%, 40%, 50%, 60% and 95% of the envelope function after previous runs. In order to diminish the effect of the starting parameter values, we discarded the first 1000 elements of each chain. Convergence of the five combined simulated chain was observed using diagnostic procedures available in *BOA* library, [25], under *R*, [15]. For each parameter we considered every 5^{th} draw and stopped at a sample of size 20 000. The hyperparameters were set so that we had a proper but very non-informative prior. For all parameters we have adopted a gamma prior distribution with shape and scale parameters equal to 0.001. From each $\hat{\beta}$ and $\hat{\theta}$ the $T_{\max}$ estimates were obtained by maximization of (3.3). Again, the Newton–Raphson method under *SAS/NLP* procedure was used. Figure 2 shows the estimated marginal posterior distribution for β, θ and $T_{\max}$.

Table 2: Maximum likelihood estimates (standard error), 95% confidence intervals, posterior means (standard deviation) and 95% credible intervals.

Parameter	MLE‡	95% Confidence Interval
β	0.8152 (0.0610)	(0.6956; 0.9348)
θ	3.8845 (0.5570)	(2.7928; 4.9762)
$T_{\max}$	3.9329	—

‡maximum likelihood estimate.

Parameter	Posterior Mean	95% Credible Interval
β	0.8102 (0.0617)	(0.6891; 0.9295)
θ	3.8714 (0.5475)	(2.8790; 4.9981)
$T_{\max}$	3.9768 (0.6800)	(2.5564; 5.2575)

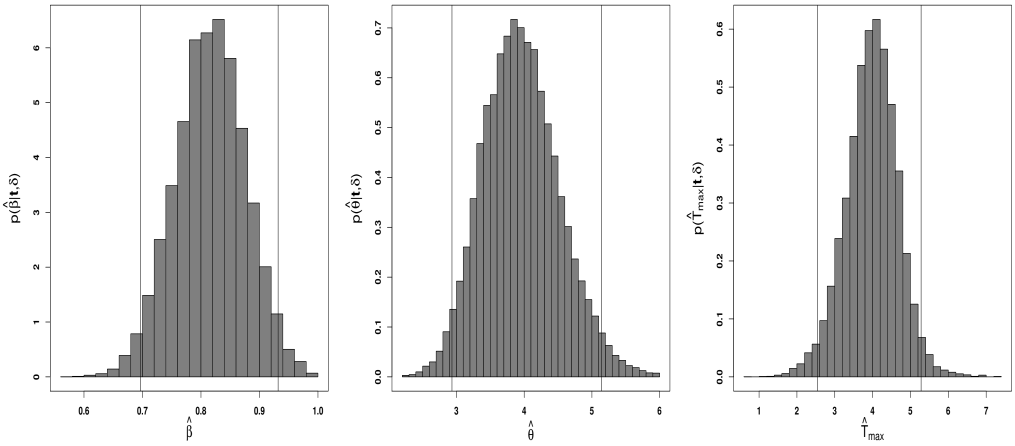

Figure 2: Estimated marginal posterior distribution for β, θ and $T_{\max}$.

5.2. An Example with a Real Data Set

As a second example, let us consider the failure data for a group of 60 electrical appliances in a life test (1000s of cycles) extracted from Lawless (2003, p. 112). Figure 3 shows the Kaplan–Meier survival curve with fit for the exponentiated Weibull distribution. We observe close agreement between the Kaplan–Meier survival curve with the exponentiated Weibull distribution. The maximum likelihood and posterior means estimates are showed in Table 3 and obtained in a similar way as considered in the simulated example.

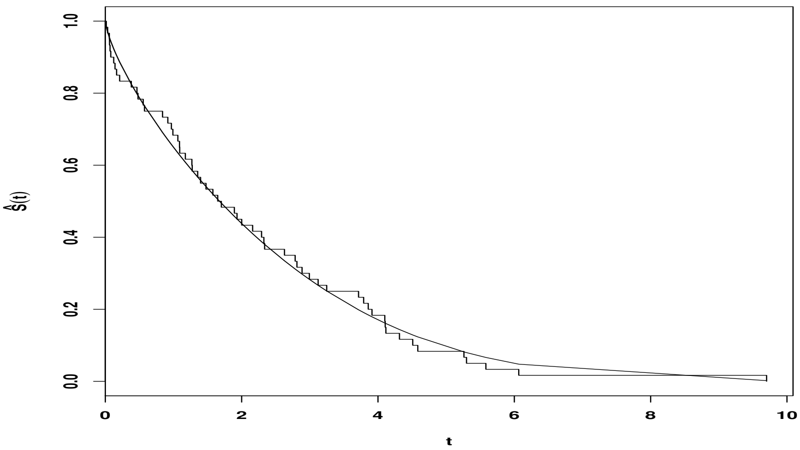

Figure 3: Fit of exponentiated Weibull distribution. (—) Kaplan–Meier survival curve and ($\cdots$) exponentiated Weibull survival function.

From the results of Table 3, we observe that $\hat{\beta} > 1$ and $\hat{\theta} < 1$, that is, we have a bathtub form for the hazard function (see Figure 1). In this situation we have interest in getting inferences for the change-point of the hazard function. This change-point is given by the minimum $T_{\min}$ of the hazard function (2.1), obtained from equation (3.4).

Table 3: Maximum likelihood estimates (standard error), 95% confidence intervals, posterior means (standard deviation) and 95% credible intervals.

Parameter	MLE[‡]	95% Confidence Interval
μ	4.1595 (0.9794)	(2.2399; 6.0791)
β	1.9599 (0.6420)	(0.7017; 3.2181)
θ	0.3717 0.1657	(0.0470; 0.6964)
$T_{\min}$	0.8865	—

[‡]maximum likelihood estimate.

Parameter	Posterior Mean	95% Credible Interval
μ	4.0491 (0.6690)	(2.8853; 5.4856)
β	1.7862 (0.3635)	(1.2142; 2.6146)
θ	0.4360 (0.1164)	(0.2470; 0.6904)
$T_{\min}$	1.0316 (0.5217)	(0.1804; 2.3321)

In Figure 4, we have the plots for the approximated marginal posterior distributions for μ, β, θ and $T_{\min}$ based on the 20000 simulated samples.

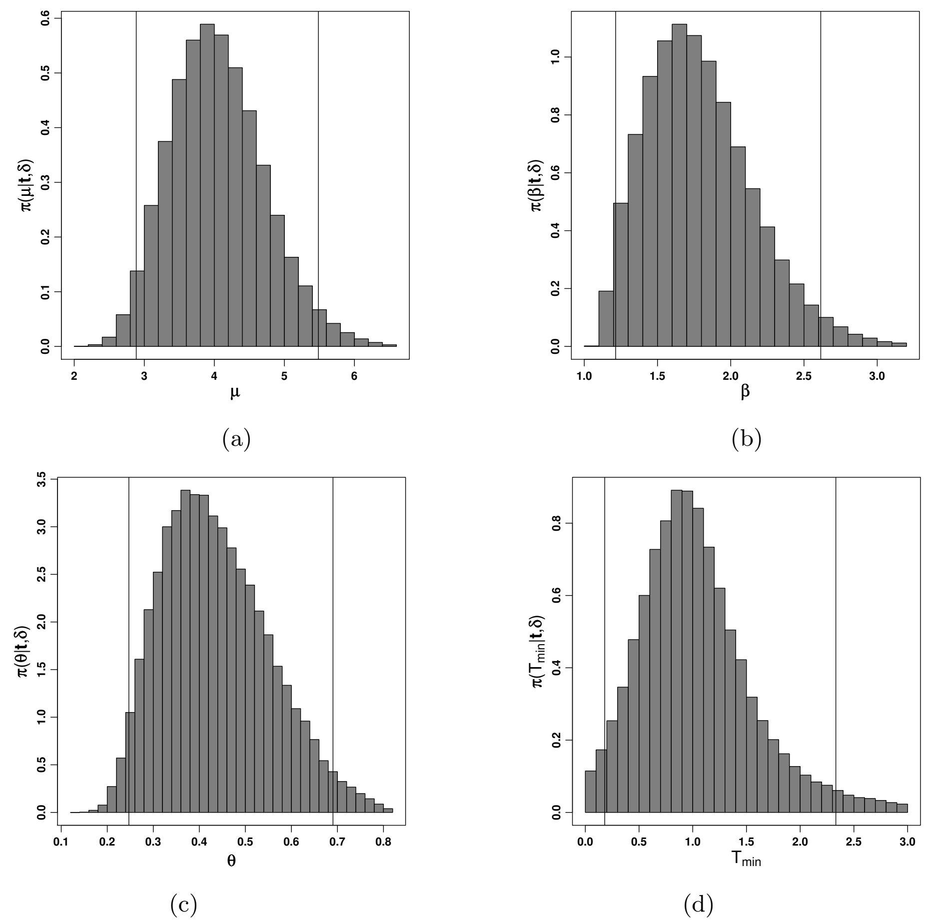

Figure 4: Marginal posterior distribution for μ, β, θ and $T_{\min}$.

6. CONCLUDING REMARKS

Inferences for the change-point of the hazard function are of great interest in lifetime studies, especially in medical or industrial applications. Assuming the exponentiated Weibull distribution we can have better fit for lifetime data since we have different shapes for the hazard function. In this situation, we do not have analytic expressions for the change-point of the hazard function (maximum if we have unimodal hazard function or minimum if we have bathtub hazard function) and we cannot use standard asymptotic classical inference methods to obtain inferences for the change-point. Using standard Markov Chain Monte Carlo simulation methods for a Bayesian analysis of the model, we get in a simple way, the posterior summaries of interest, like credible intervals for the change-points.

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