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Uncertainty-Based Analysis of Negation of Probability Distributions

Authors: Santosh Kumar Chaudhary ✉
– Department of Statistics, Central University of Jharkhand, India
skchaudhary1994@gmail.com

Pradeep Kumar Sahu
– Department of Mathematics, Indian Institute of Technology Kharagpur, India
pradeep.maths@kgpian.iitkgp.ac.in

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Abstract:

- We analyse Yager's negation operator for discrete probability distributions using Shannon entropy, varentropy and varextropy. For non-uniform distributions with $n \geq 3$, negation increases entropy and varextropy but decreases varentropy. The uniform distribution is invariant and extremises these measures. Repeated negation converges to the uniform distribution for $n \geq 3$, whereas the binary case is involutive. These results unify complementary information transformation with both first- and second-order uncertainty quantification.

Keywords:

- *extropy; entropy; varentropy; varextropy; negation.*

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- 62B10, 62D05.

✉ Corresponding author

1. INTRODUCTION

Quantifying uncertainty associated with probability distributions is a central theme in statistics, information theory, and decision sciences. Classical measures such as Shannon entropy provide a principled way to assess the average uncertainty inherent in a probabilistic system. In recent years, complementary and dual measures, including extropy and their higher-order variants such as varentropy and varextropy, have been proposed to capture additional structural and variability-related aspects of information. These developments emphasize that uncertainty is a multifaceted concept and that different representations of probability distributions may reveal distinct informational characteristics.

One important yet relatively less explored transformation of probability distributions is the negation operation. The notion of negation originates from the need to model complementary or opposing information, particularly in contexts where uncertainty arises not only from the occurrence of events but also from their non-occurrence. Yager (2015) introduced a transformation to define the negation of a probability distribution and examined its key properties. Within the framework of the Dempster–Shafer theory of evidence, it was demonstrated that, among all admissible negations, this transformation yields a negated distribution that maximizes a corresponding entropy measure. A mathematically well-founded negation for discrete probability distributions was proposed by Yager (2015). Given a probability distribution

$$P = (p_1, p_2, \ldots, p_n),$$

its negation is defined as

$$\bar{p}_i = \frac{1 - p_i}{n - 1}, \quad i = 1, 2, \ldots, n,$$

which redistributes the residual probability mass $1 - p_i$ uniformly among the remaining outcomes. This construction ensures that the negated vector $\bar{P}$ is itself a valid probability distribution and represents a state of maximum uncertainty consistent with the negation of individual events. In recent years, growing attention has been given to the study of negations of finite probability distributions, first introduced by Yager. Probability negation is defined via negators—nonincreasing functions that transform probability values element-wise—leading to different classes of negators, including distribution-independent and distribution-dependent forms. Yager's negator plays a central role in characterising linear negators and has motivated further developments, including involutive negations and their theoretical properties.

Kaur and Srivastava (2024) investigated several new properties of Yager's negation and studied its behaviour under additional information beyond the natural constraints of a probability distribution. Using probability negation, they derived the maximum entropy probability distribution (MEPD) in the presence of such constraints and showed that the existence of the MEPD, and consequently maximum entropy, strongly depends on the parameters of the imposed constraints. Numerical examples were provided to compare the MEPD with and without additional constraints. Liu and Xiao (2024) introduced an extropy-based negation method and analyzed the behavior of both the probability distribution and its associated extropy under iterative negation. Their results show that extropy increases monotonically and reaches its maximum value during the iteration process. Zhou and Deng (2023) interpreted eXtropy from the perspective of negation and extended this idea within the Dempster–Shafer theory (DST), which generalizes probability theory by capturing both event uncertainty and distributional uncertainty. They proposed a new belief entropy, termed SU entropy, defined on elements rather than mass functions, along with a novel negation of the non-specificity component in DST. Based on element-wise negation, they further introduced a belief extropy, which was shown to perform effectively in measuring uncertainty in DST.

The theoretical importance of the negation operation lies in its entropy-increasing property: the Shannon entropy of the negated distribution is greater than or equal to that of the original distribution, with equality attained for the uniform distribution. Repeated application of negation drives any initial distribution toward uniformity, which corresponds to the maximum-entropy state. From a practical perspective, negation plays a significant role in uncertainty modelling, information fusion, decision-making under incomplete evidence, and fuzzy and belief-based systems, where complementary or opposite information must be systematically incorporated.

While existing studies have primarily focused on the interaction between negation and entropy or extropy, the effect of negation on higher-order information measures remains largely unexplored. Measures such as varentropy and varextropy quantify the variability or dispersion of information content and thus provide a deeper characterisation of uncertainty beyond its average level. Understanding how these measures behave under negation is essential for developing a more comprehensive theory of complementary uncertainty representations.

Motivated by these considerations, the present paper investigates the interplay among probability distributions, negation operations, and information-theoretic measures within a unified framework. Specifically, we analyze the effect of Yager's negation on Shannon entropy, extropy, varentropy, and varextropy. We theoretically establish that negation increases all these measures, thereby extending known results for entropy and extropy to their second-order counterparts. These findings deepen the understanding of how complementary transformations of probability distributions influence both the magnitude and variability of information and provide new insights into negation-based uncertainty analysis.

Negation is a pivotal method in information processing, particularly when dealing with uncertain or incomplete data. In many situations, attention is primarily focused on the positive aspects of information, while the negative components are often overlooked. However, analyzing uncertainty from a negative perspective can reveal insights that may not be readily apparent when considering only the positive side. In certain cases, information that is difficult to interpret from a positive viewpoint becomes more accessible through negation. A more accurate and comprehensive understanding of a problem can therefore be achieved by jointly considering both positive and negative aspects of information.

In this context, we address the problem of determining the negation of a probability distribution and examine how negation affects key measures associated with probability distributions. In particular, our focus is on varentropy and varextropy, which are important metrics for understanding changes in the characteristics of a distribution under negation. These measures provide insight into how negation influences uncertainty and how it can be effectively used to refine the analysis of uncertain data.

Negation methods are widely used in evidence theory, a framework developed for reasoning and decision-making under uncertainty. Within this framework, negation techniques are employed to modify models based on uncertain or incomplete information in order to better capture underlying uncertainty. Among the various approaches to negation, entropy-based methods play a crucial role. For example, Yager (2015) introduced a technique for negating probability distributions based on Gini entropy and also explored the role of Tsallis entropy in this context. These entropy-based approaches provide a systematic way to process uncertainty and to understand how negation alters the informational structure of probability distributions.

In addition to entropy, extropy has recently attracted increasing attention as a complementary measure of uncertainty. Extropy is often regarded as the dual counterpart of entropy and serves a similar purpose in quantifying uncertainty. According to Lad et al. (2015), the probability distribution that maximises extropy is the uniform distribution. Although extropy is mathematically related to entropy, it is expressed in a different form and offers an alternative perspective on uncertainty. Owing to its conceptual and analytical advantages, extropy has become an important tool for studying the effects of negation on probability distributions.

Varentropy is a measure that quantifies the variability in the information content of a random vector and is invariant under affine transformations. Together with varextropy, it provides a richer description of uncertainty by capturing fluctuations in information content rather than only its average value.

1.1. Shannon entropy

The entropy of a discrete probability distribution $P = \{p_1, \ldots, p_n\}$ is defined as

$$H(P) = -\sum_{i=1}^{n} p_i \ln p_i.$$

1.2. Varentropy

The varentropy of a discrete probability distribution $P = \{p_1, \ldots, p_n\}$ is defined as (see Bobkov and Madiman, 2011; Kontoyiannis and Verdú, 2014; Arıkan, 2016; Di Crescenzo and Paolillo, 2021; Maadani et al., 2021)

$$VH(P) = \sum_{i=1}^{n} p_i (\ln p_i)^2 - \left(\sum_{i=1}^{n} p_i \ln p_i\right)^2.$$

Varentropy serves as a measure of the variability in the information content.

1.3. Extropy

Lad et al. (2015) introduced the concept of extropy as a complement to Shannon entropy. The extropy of a discrete probability distribution $P = \{p_1, \ldots, p_n\}$ is defined as

$$J(P) = -\sum_{i=1}^{n}(1 - p_i)\ln(1 - p_i).$$

1.4. Varextropy

The varextropy of a discrete probability distribution $P = \{p_1, \ldots, p_n\}$ is defined as (see Vaselabadi et al., 2021; Goodarzi, 2022; Zaid et al., 2022)

$$VJ(P) = \sum_{i=1}^{n}(1 - p_i)\left(\ln(1 - p_i)\right)^2 - \left(\sum_{i=1}^{n}(1 - p_i)\ln(1 - p_i)\right)^2.$$

Varextropy also serves as a measure of the variability in the information content.

In this paper, we study the negation of a probability distribution based on Shannon entropy, varentropy and varextropy. This approach provides a new perspective on how negation can be applied to probability distributions and offers additional tools for analysing uncertainty. By incorporating these advanced measures, the proposed framework enables a deeper understanding of how negation influences both the magnitude and variability of information, thereby enhancing the analysis of uncertain data. Section 2 defines the negation method and the negation of uncertainty measures. Section 3 deals with some examples. Section 4 presents some results. The conclusion is given in Section 5.

2. NEGATION OF A DISCRETE PROBABILITY DISTRIBUTION

Yager (2015) investigated the negation of a given probability distribution using the entropy function. Yager (2015) explored how the knowledge within the negation of a probability distribution can be represented. He proposed a transformation method to derive the negation of a probability distribution and examined its properties. By applying the Dempster–Shafer theory of evidence, Yager (2015) demonstrated that among all possible negations, the one suggested in his work exhibits the maximal type of entropy. While there are many different measures of entropy, Yager (2015) used the following to measure the entropy of a probability distribution,

$$H_1(P) = \sum_{i=1}^{n}(1 - p_i)p_i = 1 - \sum_{i=1}^{n}p_i^2.$$

Yager (2015) selected this form of entropy measure instead of the classic Shannon entropy due to the simplicity of calculation it offers, as it does not involve logarithms. Yager

(2015) defined negation of entropy as

$$H_1(\bar{P}) = \sum_{i=1}^{n}(1-\bar{p}_i)\bar{p}_i = 1 - \sum_{i=1}^{n}\bar{p}_i{}^2.$$

Liu and Xiao (2024) investigated the negation of a given probability distribution using the extropy function

$$J_1(P) = -\sum_{i=1}^{n}(1-p_i)\ln(1-p_i).$$

Liu and Xiao (2024) defined negation of extropy as

$$J_1(\bar{P}) = \sum_{i=1}^{n}(1-\bar{p}_i)\ln(1-\bar{p}_i).$$

They demonstrated that through repeated negation of the probability, both the probability distribution and its associated extropy converge to stable values.

2.1. Negation method

Yager (2015) considered the problem of finding the negation of a probability distribution and suggested following a negation of a discrete probability distribution. Assume $X = \{x_1, \ldots, x_n\}$ and $P = \{p_1, \ldots, p_n\}$ is the probability distribution of X such that $\sum_{i=1}^{n} p_i = 1$, and $p_i \in [0, 1]$. Let $\bar{P} = \{\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n\}$ represent the negation of the probability distribution P, where the negation of the probability is given by Yager (2015) and $\bar{p}_i = \frac{1-p_i}{n-1}$. $\bar{P}$ is also a probability distribution (Yager, 2015) since $\sum_{i=1}^{n} \bar{p}_i = 1$ and $\bar{p}_i \in [0, 1]$. The inverse is irreversible in general, which means $p_i \neq \bar{\bar{p}}_i$ for $n = 3, 4, \ldots$, but $p_i = \bar{\bar{p}}_i$ when $n = 2$ (Yager, 2015). Moreover, $0 \leq \bar{p}_i \leq \frac{1}{n-1}$, $i = 1, 2, \ldots, n$ (Liu and Xiao, 2024). The fact that the operation is irreversible in general adds an interesting aspect to the analysis of uncertainty. If P is uniform, i.e., $p_i = \frac{1}{n}$ for all i, then

$$\bar{p}_i = \frac{1 - 1/n}{n-1} = \frac{1}{n},$$

hence the uniform distribution is invariant under negation. The negation of probability distributions may have applications in risk analysis, machine learning, data science, information theory, and decision theory.

2.2. Negation with Shannon entropy

The negation of entropy

$$H(P) = -\sum_{i=1}^{n} p_i \ln p_i$$

is given as

$$H(\bar{P}) = -\sum_{i=1}^{n} \bar{p}_i \ln \bar{p}_i.$$

2.3. Negation with extropy

The negation of extropy

$$J(P) = -\sum_{i=1}^{n}(1-p_i)\ln(1-p_i)$$

is given as

$$J(\bar{P}) = -\sum_{i=1}^{n}(1-\bar{p}_i)\ln(1-\bar{p}_i).$$

2.4. Negation with varentropy

The negation of varentropy

$$VH(P) = \sum_{i=1}^{n} p_i\,(\ln(p_i))^2 - \left(\sum_{i=1}^{n} p_i\ln(p_i)\right)^2$$

is given as

$$VH(\bar{P}) = \sum_{i=1}^{n} \bar{p}_i\,(\ln(\bar{p}_i))^2 - \left(\sum_{i=1}^{n} \bar{p}_i\ln(\bar{p}_i)\right)^2.$$

2.5. Negation with varextropy

The negation of varextropy

$$VJ(P) = \sum_{i=1}^{n}(1-p_i)\,(\ln((1-p_i)))^2 - \left(\sum_{i=1}^{n}(1-p_i)\ln((1-p_i))\right)^2$$

is given as

$$VJ(\bar{P}) = \sum_{i=1}^{n}(1-\bar{p}_i)\,(\ln((1-\bar{p}_i)))^2 - \left(\sum_{i=1}^{n}(1-\bar{p}_i)\ln((1-\bar{p}_i))\right)^2.$$

3. EXAMPLES

Example 3.1. Let $n = 2$, $X = \{x_1, x_2\}$, and $P = \{p_1, p_2\}$. Let $p_1 = p_2 = \frac{1}{2}$, then $H(P) = H(\bar{P}) = \ln 2$, $J(P) = J(\bar{P}) = \ln 2$, $VH(P) = VH(\bar{P}) = 0$ and $VJ(P) = VJ(\bar{P}) = 0$.

Example 3.2. Let $n=2$, $X=\{x_1,x_2\}$, and $P=\{p_1,p_2\}$. Let $p_1=\frac{2}{5}, p_2=\frac{3}{5}$, then $H(P)=H(\bar{P})=0.6730,\ J(P)=J(\bar{P})=0.6730,\ VH(P)=VH(\bar{P})=0.03946,\ VJ(P)=VJ(\bar{P})=0.03946$.

Example 3.3. Let $n=2$, $X=\{x_1,x_2\}$, and $P=\{p_1,p_2\}$. Let $p_1=\frac{1}{10}, p_2=\frac{9}{10}$, then $H(P)=H(\bar{P})=0.3251,\ J(P)=J(\bar{P})=0.3251,\ VH(P)=VH(\bar{P})=0.4345,\ VJ(P)=VJ(\bar{P})=0.4345$.

When $n=2$, entropy, extropy, varentropy, and varextropy will not change value after probability distribution negation iteration. As the number of iterations increases, the entropy, varentropy and varextropy still remain unchanged for $n=2$. That is, $H(P)=H(\bar{P})=H(\bar{\bar{P}})=\cdots$, $VH(P)=VH(\bar{P})=VH(\bar{\bar{P}})=\cdots$ and $VJ(P)=VJ(\bar{P})=VJ(\bar{\bar{P}})=\cdots$ for $n=2$.

Example 3.4. For n equally likely outcomes (uniform distribution), each p_i is given by,

$$p_i=\frac{1}{n}\quad \text{for all}\quad i=1,2,\ldots,n;\ n>1.$$

For the negated distribution $\bar{P}$, the probabilities are,

$$\bar{p}_i=\frac{1-p_i}{n-1}=\frac{1-\frac{1}{n}}{n-1}=\frac{n-1}{n(n-1)}=\frac{1}{n}, \text{for all}\quad i=1,2,\ldots,n;\ n>1.$$

Then,

$$H(P)=-\sum_{i=1}^{n}p_i\ln(p_i)=-n\left(\frac{1}{n}\right)\ln\left(\frac{1}{n}\right)=\ln(n),\ H(\bar{P})=\ln(n),$$

$$J(P)=J(\bar{P})=-(n-1)\ln\left(\frac{n-1}{n}\right),$$

$$VH(P)=VH(\bar{P})=(\ln n)^2-(-\ln n)^2=(\ln n)^2-(\ln n)^2=0$$

$$\text{and}\ \ VJ(P)=VJ(\bar{P})=n\left(1-\frac{1}{n}\right)\left(\ln\left(1-\frac{1}{n}\right)\right)^2-\left(n\left(1-\frac{1}{n}\right)\ln\left(1-\frac{1}{n}\right)\right)^2.$$

We observe that as n increases, $H(P)$ increase and $VJ(P)$ decreses with respect to n when $p_i=\frac{1}{n}$ for all $i=1,2,\ldots,n;\ n>1$.

Example 3.5. Let $n=3$, $X=\{x_1,x_2,x_3\}$, and $P=\{p_1,p_2,p_3\}$. Let $p_1=\frac{1}{3}, p_2=\frac{1}{3},\ p_3=\frac{1}{3}$, then $H(P)=H(\bar{P})=\ln 3,\ VH(P)=VH(\bar{P})=0$, and $VJ(P)=VJ(\bar{P})=0.0837$.

Example 3.6. Let $n=3$, $X=\{x_1,x_2,x_3\}$, and $P=\left\{\frac{1}{10},\frac{3}{10},\frac{6}{10}\right\}$. Since $n=3$, the negated distribution is $\bar{p}_i=\frac{1-p_i}{2}$. Thus, $\bar{P}=\left(\frac{9}{20},\frac{7}{20},\frac{4}{20}\right)=(0.45,\ 0.35,\ 0.20)$.

$$H(P)=0.8979,\ VH(P)=0.3084, VJ(P)=0.0390$$

$$H(\bar{P})=1.0487,\ VH(\bar{P})=0.0363, VJ(\bar{P})=0.2100.$$

Example 3.7. Let the probability distribution be $P = \{0.4, 0.3, 0.2, 0.1\}$. Since $n = 4$, the negated probabilities are $\bar{p}_i = \frac{1-p_i}{3}$. Thus,

$$\bar{P} = \left(\frac{0.6}{3}, \frac{0.7}{3}, \frac{0.8}{3}, \frac{0.9}{3}\right) = (0.2000,\ 0.2333,\ 0.2667,\ 0.3000).$$

Further, $\bar{\bar{P}} = (0.2667,\ 0.2556,\ 0.2444,\ 0.2333)$, $\bar{\bar{\bar{P}}} = (0.2444,\ 0.2481,\ 0.2519,\ 0.2556)$. We have

$$H(P) = 1.2799,\ VH(P) = 0.2546,\ VJ(P) = 0.0123.$$

Corresponding values after the first negation are

$$H(\bar{P}) = 1.3801,\ VH(\bar{P}) = 0.0159,\ VJ(\bar{P}) = 0.1202.$$

Corresponding values after the second negation are

$$H(\bar{\bar{P}}) = 1.3853,\ VH(\bar{\bar{P}}) = 0.0018,\ VJ(\bar{\bar{P}}) = 0.1506.$$

Corresponding values after the third negation are

$$H(\bar{\bar{\bar{P}}}) = 1.3861,\ VH(\bar{\bar{\bar{P}}}) = 0.0002,\ VJ(\bar{\bar{\bar{P}}}) = 0.1594.$$

Example 3.8. Let $X = \{x_1, x_2, x_3\}$ and the probability distribution $P = \{0.6, 0.3, 0.1\}$. The negated distribution is $\bar{P} = \{0.2, 0.35, 0.45\}$. Also, $\bar{\bar{P}} = (0.4,\ 0.325,\ 0.275)$, $\bar{\bar{\bar{P}}} = (0.3,\ 0.3375,\ 0.3625)$, and $\bar{\bar{\bar{\bar{P}}}} = (0.35,\ 0.33125,\ 0.31875)$. We have

$$H(P) = 0.8979,\ VH(P) = 0.3084,\ VJ(P) = 0.0390.$$

Corresponding values after the first negation are

$$H(\bar{P}) = 1.0487,\ VH(\bar{P}) = 0.0363,\ VJ(\bar{P}) = 0.2100.$$

Corresponding values after the second negation are

$$H(\bar{\bar{P}}) = 1.0861,\ VH(\bar{\bar{P}}) = 0.0148, VJ(\bar{\bar{P}}) = 0.2687.$$

Corresponding values after the third negation are

$$H(\bar{\bar{\bar{P}}}) = 1.0954,\ VH(\bar{\bar{\bar{P}}}) = 0.0037,\ VJ(\bar{\bar{\bar{P}}}) = 0.2850.$$

Corresponding values after the fourth negation are

$$H(\bar{\bar{\bar{\bar{P}}}}) = 1.0977,\ VH(\bar{\bar{\bar{\bar{P}}}}) = 0.0009,\ VJ(\bar{\bar{\bar{\bar{P}}}}) = 0.2889.$$

Example 3.9. Let $P = (0.7,\, 0.2,\, 0.1)$. Since $n = 3$, $\bar{p}_i = \frac{1-p_i}{2}$. Thus,

$$\bar{P} = \left(\frac{1-0.7}{2}, \frac{1-0.2}{2}, \frac{1-0.1}{2} \right) = (0.15,\, 0.40,\, 0.45).$$

Observe that the original distribution is highly skewed, whereas the negated distribution is more balanced. We got $H(P) = 0.801$, and $H(\bar{P}) = 1.011$. Hence, entropy increases after negation. Applying negation again, $\bar{\bar{P}} \neq P$, which confirms irreversibility for $n = 3$.

Iteration k	$\mathbf{H(P^{(k)})}$	$\mathbf{VH(P^{(k)})}$	$\mathbf{VJ(P^{(k)})}$
0	0.8979	0.3084	0.0390
1	1.0487	0.0363	0.2100
2	1.0860	0.0148	0.2687
3	1.0954	0.0037	0.2850
4	1.0977	0.0009	0.2889

Table 1: Convergence of H, VH, and VJ under repeated Yager negation ($n = 3,\ P^{(0)} = (0.1, 0.3, 0.6)$).

Table 1 illustrates the behaviour of three uncertainty measures—entropy (H), varentropy (VH), and varextropy (VJ)—under repeated applications of Yager's negation operator for $n = 3$ with initial distribution $P^{(0)} = (0.1, 0.3, 0.6)$.

The table shows the values of each measure at iterations $k = 0$ to $k = 4$. It is evident that:

(i) Entropy H increases toward the maximum value $\ln 3 = 1.0986$, reflecting the distribution's progression toward uniformity.

(ii) Varentropy VH diminishes rapidly toward zero, demonstrating that fluctuations in the probability distribution vanish as it becomes uniform.

(iii) Varextropy VJ increases and stabilises, showing that the variance of complement log-probabilities reaches a steady state under repeated negation.

Also, note that for any non-uniform distribution P, $H(\bar{P}) \geq H(P)$. This shows that negation acts as an uncertainty-increasing transformation.

4. SOME RESULTS

Theorem 4.1. *Assume the event space* $X = \{x_1, x_2, \ldots, x_n\}$ *and the probability distribution* $P = \{p_1, p_2, \ldots, p_n\}$, $\bar{P}$ *represents the inverse of P, then* $H(\bar{P}) \geq H(P)$.

Proof: We have seen in Section 3 that $H(P) = H(\bar{P})$ for $n = 2$. For $n \geq 3$, the difference

$$\begin{aligned} Y_1 = H(P) - H(\bar{P}) &= -\sum_{i=1}^{n} p_i \ln p_i + \sum_{i=1}^{n} \bar{p}_i \ln \bar{p}_i \\ &= -\sum_{i=1}^{n} p_i \ln p_i + \sum_{i=1}^{n} \left(\frac{1-p_i}{n-1}\right) \ln \frac{1-p_i}{n-1}. \end{aligned} \tag{4.1}$$

The Lagrange function under equation (4.1) is

$$T_1 = -\sum_{i=1}^{n} p_i \ln p_i + \sum_{i=1}^{n} \left(\frac{1-p_i}{n-1}\right) \ln \frac{1-p_i}{n-1} + \lambda \left(\sum_{i=1}^{n} p_i - 1\right). \tag{4.2}$$

The partial derivative of T_1 with respect to p_i is

$$\frac{\partial T_1}{\partial p_i} = -\ln(p_i) - \frac{1}{n-1} \ln\left(\frac{1-p_i}{n-1}\right) - \frac{n}{n-1} + \lambda, \ \forall\ i = 1, 2, \ldots, n. \tag{4.3}$$

and

$$\frac{\partial T_1}{\partial \lambda} = \sum_{i=1}^{n} p_i - 1. \tag{4.4}$$

Lets solve $\frac{\partial T_1}{\partial p_i} = 0$ and $\frac{\partial T_1}{\partial \lambda} = 0$ to get stationary points. Equation (4.3) can be written as

$$-\ln(p_i) - k_1 \ln(1-p_i) = k_2 \quad \text{for all} \quad i = 1, 2, ..., n \quad \text{and} \quad \sum_{i=1}^{n} p_i = 1, \tag{4.5}$$

where $k_1 = \frac{1}{n-1}$ and $k_2 = \frac{n}{n-1} - \frac{ln(n-1)}{n-1} - \lambda$. Hence, T_1 has maximum value 0 when $p_i = 1/n$, therefore, $Y_1 \leq 0$. That is, $H(\bar{P}) \geq H(P)$. □

Theorem 4.2. *Assume the event space $X = \{x_1, x_2, \ldots x_n\}$ and the probability distribution $P = \{p_1, p_2, \ldots, p_n\}$, $\bar{P}$ represents the inverse of P, then $VH(\bar{P}) \leq VH(P)$.*

Proof: We have seen in Section 3 that $VH(P) = VH(\bar{P})$ for $n = 2$. For $n \geq 3$, the difference

$$\begin{aligned} Y_2 =& VH(P) - VH(\bar{P}) \\ =& \left(\sum_{i=1}^{n} p_i (\ln(p_i))^2 - \left(\sum_{i=1}^{n} p_i \ln(p_i)\right)^2\right) \\ &- \left(\sum_{i=1}^{n} \left(\frac{1-p_i}{n-1}\right) \left(\ln(\left(\frac{1-p_i}{n-1}\right)\right)^2 - \left(\sum_{i=1}^{n} \left(\frac{1-p_i}{n-1}\right) \ln\left(\frac{1-p_i}{n-1}\right)\right)^2\right). \end{aligned} \tag{4.6}$$

The Lagrange function under equation 4.6 is

$$T_2 = \left(\sum_{i=1}^{n} p_i \left(\ln(p_i)\right)^2 - \left(\sum_{i=1}^{n} p_i \ln(p_i) \right)^2 \right)$$

$$(4.7) \quad - \sum_{i=1}^{n} \left(\frac{1-p_i}{n-1} \right) \left(\ln(\left(\frac{1-p_i}{n-1} \right) \right)^2 + \left(\sum_{i=1}^{n} \left(\frac{1-p_i}{n-1} \right) \ln \left(\frac{1-p_i}{n-1} \right) \right)^2 + \lambda \left(\sum_{i=1}^{n} p_i - 1 \right).$$

The partial derivative of T_2 with respect to p_i is

$$\frac{\partial T_2}{\partial p_i} = (\ln(p_i))^2 + 2\ln(p_i) - 2 \left(\sum_{i=1}^{n} p_i \ln(p_i) \right) (\ln(p_i) + 1)$$
$$- \left[\frac{1}{n-1} \left(\ln \left(\frac{1-p_i}{n-1} \right) \right)^2 + 2 \ln \left(\frac{1-p_i}{n-1} \right) \left(\frac{1}{n-1} \right) \left(- \frac{1}{1-p_i} \right) \right]$$
$$(4.8) \quad +2 \left(\sum_{i=1}^{n} \left(\frac{1-p_i}{n-1} \right) \ln \left(\frac{1-p_i}{n-1} \right) \right) \left(- \frac{1}{n-1} \ln \left(\frac{1-p_i}{n-1} \right) + \frac{1-p_i}{(n-1)(1-p_i)} \right) + \lambda$$

and

$$(4.9) \quad \frac{\partial T_2}{\partial \lambda} = \sum_{i=1}^{n} p_i - 1.$$

Putting $\frac{\partial T_2}{\partial p_i} = 0 \quad \forall \ i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} p_i = 1$ and since T_2 involves both variance like terms (which are minimized when p_i are equal) and logarithmic terms (which are also minimized when p_i are uniform), the solution to this minimization problem under the constraint $\sum_{i=1}^{n} p_i = 1$ is the uniform distribution $p_i = \frac{1}{n}$ for $i = 1, 2, \ldots, n$. This is the only point where T_2 achieves its minimum because of the concavity of the function. Since the function is concave over a convex set and the minimum occurs at $p_i = \frac{1}{n}$, we can say that $p_i = \frac{1}{n}$ for $i = 1, 2, \ldots, n$. is the unique solution.

Hence, T_2 has minimum value 0 when $p_i = \frac{1}{n}$ therefore, $Y_2 \geq 0$. That is, $VH(\bar{P}) \leq VH(P)$. □

Theorem 4.3. *Assume the event space $X = \{x_1, x_2, \ldots, x_n\}$ and the probability distribution $P = \{p_1, p_2, \ldots, p_n\}$, $\bar{P}$ represents the inverse of P, then $VJ(\bar{P}) \geq VJ(P)$.*

Proof: We know from Section 3 that for $n = 2$, the varextropy of P and $\bar{P}$ are equal, i.e., $VJ(P) = VJ(\bar{P})$. Thus, inequality is present in this case.

To prove the inequality $VJ(\bar{P}) \geq VJ(P)$, we consider the difference between the varextropy of the probability distribution P and its negation $\bar{P}$.

We define the difference between varextropy as:

$$Y_3 = VJ(P) - VJ(\bar{P}).$$

Expanding the terms, we have:

$$Y_3 = \left(\sum_{i=1}^{n}(1-p_i)\left(\ln\left((1-p_i)\right)\right)^2 - \left(\sum_{i=1}^{n}(1-p_i)\ln\left((1-p_i)\right)\right)^2\right)$$

$$-\left(\sum_{i=1}^{n}\left(1-\frac{1-p_i}{n-1}\right)\left(\ln\left(\left(1-\frac{1-p_i}{n-1}\right)\right)\right)^2 - \left(\sum_{i=1}^{n}\left(1-\frac{1-p_i}{n-1}\right)\ln\left(\left(1-\frac{1-p_i}{n-1}\right)\right)\right)^2\right).$$

This difference represents the change in varextropy after negating the probability distribution.

To analyse this further, we introduce a Lagrange multiplier function to optimise the varextropy expression. The Lagrange function T_3 is given as:

$$T_3 = \left(\sum_{i=1}^{n}(1-p_i)\left(\ln\left((1-p_i)\right)\right)^2 - \left(\sum_{i=1}^{n}(1-p_i)\ln\left((1-p_i)\right)\right)^2\right)$$

$$-\sum_{i=1}^{n}\left(1-\frac{1-p_i}{n-1}\right)\left(\ln\left(\left(1-\frac{1-p_i}{n-1}\right)\right)\right)^2 + \left(\sum_{i=1}^{n}\left(1-\frac{1-p_i}{n-1}\right)\ln\left(\left(1-\frac{1-p_i}{n-1}\right)\right)\right)^2$$

$$+\lambda\left(\sum_{i=1}^{n}p_i-1\right).$$

To find the optimal values of p_i, we compute the partial derivatives of T_3 with respect to p_i, and with respect to λ:

The partial derivative with respect to p_i is:

$$\frac{\partial T_3}{\partial p_i} = 2\ln(1-p_i)\left(-\frac{1}{1-p_i}\right)$$

$$+2\sum_{i=1}^{n}\left(1-\frac{1-p_i}{n-1}\right)\ln\left(\left(1-\frac{1-p_i}{n-1}\right)\right)\left(-\frac{1}{n-1}\ln\left(\left(1-\frac{1-p_i}{n-1}\right)\right)+\frac{1-p_i}{(n-1)(1-p_i)}\right)$$

and similar terms for other partial derivatives.

The partial derivative with respect to λ is:

$$\frac{\partial T_3}{\partial \lambda} = \sum_{i=1}^{n}p_i - 1.$$

Setting the partial derivatives $\frac{\partial T_3}{\partial p_i} = 0$ for all i and enforcing the constraint $\sum_{i=1}^{n} p_i = 1$ gives $p_i = \frac{1}{n}$ for all $i = 1, 2, \ldots, n$.

The solution $p_i = 1/n$ for all i is the unique maximiser for T_3 under the constraint $\sum_{i=1}^{n} p_i = 1$ because T_3 is a strictly concave and symmetric function on the probability simplex. Strict concavity ensures that any local maximum is the global and only maximum, and symmetry means the uniform distribution is the only critical point satisfying the constraint. Therefore, the maximum value of T_3 is zero, uniquely achieved at $p_i = 1/n$ for all i.

When $p_i = \frac{1}{n}$ for all i, the value of T_3 is maximized and the difference Y_3 is less than or equal to zero:

$$Y_3 \leq 0.$$

This implies that:

$$VJ(\bar{P}) \geq VJ(P).$$

□

Thus, we have proven that the varextropy of the negated probability distribution $\bar{P}$ is greater than or equal to the varextropy of the original distribution P.

Theorem 4.4. *When $X = \{x_1, x_2, \ldots, x_n\}$, and the probability distribution is $P = \{p_1, p_2, \ldots, p_n\}$, and $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$ the value of entropy increases with the increase of the size of X, and the maximum value of entropy tends to $+\infty$ as n tends to $+\infty$.*

Proof: Let $P = \{p_1, p_2, \ldots, p_n\}$ be a discrete probability distribution, and the Shannon entropy is given by:

$$H(P) = -\sum_{i=1}^{n} p_i \ln p_i$$

For a uniform probability distribution, the Shannon entropy is:

$$H(P) = \ln n$$

which increases with an increase in sample size n. Therefore, the limit of Shannon entropy as $n \to \infty$ is:

$$\lim_{n\to\infty} H(P) = \infty.$$

□

Theorem 4.5. *When $X = \{x_1, x_2, \ldots x_n\}$, and the probability distribution is $P = \{p_1, p_2, \ldots, p_n\}$, and $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$ the value of varentropy remains same with the increase of the size of X, and the value of varentropy is zero.*

Proof: Let $P = \{p_1, p_2, \ldots, p_n\}$ be a discrete probability distribution, and the varentropy $VH(P)$ is given by:

$$VH(P) = \sum_{i=1}^{n} p_i \left(\ln(p_i)\right)^2 - \left(\sum_{i=1}^{n} p_i \ln(p_i)\right)^2.$$

For a uniform distribution, where $p_i = \frac{1}{n}$, we have:

$$VH(P) = \sum_{i=1}^{n} \frac{1}{n} \left(\ln\left(\frac{1}{n}\right)\right)^2 - \left(\sum_{i=1}^{n} \frac{1}{n} \ln\left(\frac{1}{n}\right)\right)^2$$

$$VH(P) = (\ln n)^2 - (\ln n)^2 = 0.$$

For a uniform distribution, the value of varentropy is always 0, so:

$$\lim_{n\to\infty} VH(P) = 0.$$

Thus, varentropy does not increase with n and remains 0 for a uniform probability distribution. □

Theorem 4.6. *When $X = \{x_1, x_2, \ldots, x_n\}$ and the probability distribution is $P = \{p_1, p_2, \ldots, p_n\}$, with $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$, the value of varextropy decreases with the increase in the size of X, and the minimum value of varextropy has a limit -1.*

Proof: The varextropy $VJ(P)$ for a discrete probability distribution is given by:

$$VJ(P) = \sum_{i=1}^{n}(1-p_i)\left(\ln((1-p_i))\right)^2 - \left(\sum_{i=1}^{n}(1-p_i)\ln((1-p_i))\right)^2.$$

For the uniform distribution, where $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$, we substitute $p_i = \frac{1}{n}$ and we get,

$$\begin{aligned} VJ(P) &= \sum_{i=1}^{n}\left(1-\frac{1}{n}\right)\left(\ln\left(1-\frac{1}{n}\right)\right)^2 - \left(\sum_{i=1}^{n}\left(1-\frac{1}{n}\right)\ln\left(1-\frac{1}{n}\right)\right)^2 \\ &= n\cdot\left(1-\frac{1}{n}\right)\left(\ln\left(1-\frac{1}{n}\right)\right)^2 - \left(n\cdot\left(1-\frac{1}{n}\right)\ln\left(1-\frac{1}{n}\right)\right)^2. \end{aligned} \tag{4.10}$$

Therefore,

$$VJ(P) \to -1 \quad \text{as} \quad n \to \infty.$$

Thus, the value of varextropy approaches -1 as n increases. □

Theorem 4.7. *Assume the $X = \{x_1, x_2, \ldots x_n\}$, when the probability distribution satisfies $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$, the corresponding entropy of the probability distribution after negation is maximised.*

Proof: The Shannon entropy for a probability distribution $P = \{p_1, p_2, \ldots, p_n\}$ is given by:

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i.$$

The negation of the probability distribution P, denoted as $\bar{P} = \{\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n\}$, is defined as:

$$\bar{p}_i = \frac{1-p_i}{n-1}.$$

The Shannon entropy of the negated distribution $\bar{P}$ is:

$$H(\bar{P}) = -\sum_{i=1}^{n} \bar{p}_i \log \bar{p}_i = -\sum_{i=1}^{n} \frac{1-p_i}{n-1} \log\left(\frac{1-p_i}{n-1}\right).$$

We now proceed to maximise $H(\bar{P})$ using the Lagrange multiplier method. The Lagrangian for this problem is:

$$\mathcal{L}(p_1, p_2, \dots, p_n, \lambda) = -\sum_{i=1}^{n} \frac{1-p_i}{n-1} \log\left(\frac{1-p_i}{n-1}\right) + \lambda\left(\sum_{i=1}^{n} p_i - 1\right).$$

Taking the derivative with respect to p_i and setting it to zero:

$$\frac{\partial \mathcal{L}}{\partial p_i} = 0 \quad \Rightarrow \quad \log\left(\frac{1-p_i}{n-1}\right) = -1 - \lambda(n-1).$$

Solving for p_i:

$$p_i = 1 - (n-1)e^{-1-\lambda(n-1)}. \tag{4.11}$$

Applying the condition $\sum_{i=1}^{n} p_i = 1$, we solve for λ and we get,

$$\lambda = \frac{\ln n - 1}{n-1}.$$

Substituting this back into the equation 4.11 for p_i, we find the optimal values of $p_i = \frac{1}{n}, \ \forall \ i = 1, 2, 3, \dots, n$ that maximize the $H(\bar{P})$. The maximum value of $H(\bar{P})$ is,

$$H(\bar{P}) = \ln n. \tag{4.12}$$

□

Theorem 4.8. *Assume the $X = \{x_1, x_2, \dots, x_n\}$, when the probability distribution satisfies $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$, the corresponding varentropy of the probability distribution after negation is minimised.*

Proof: The varentropy of a discrete probability distribution $P = \{p_1, p_2, \dots, p_n\}$ is defined as:

$$VH(P) = \sum_{i=1}^{n} p_i \left(\ln(p_i)\right)^2 - \left(\sum_{i=1}^{n} p_i \ln(p_i)\right)^2.$$

The negation of the probability distribution P, denoted as $\bar{P} = \{\bar{p}_1, \bar{p}_2, \dots, \bar{p}_n\}$, is given by:

$$\bar{p}_i = \frac{1-p_i}{n-1}.$$

The negated varentropy $VH(\bar{P})$ is then:

$$VH(\bar{P}) = \sum_{i=1}^{n} \bar{p}_i \left(\ln(\bar{p}_i)\right)^2 - \left(\sum_{i=1}^{n} \bar{p}_i \ln(\bar{p}_i)\right)^2.$$

Substituting $\bar{p}_i = \frac{1-p_i}{n-1}$ into this formula:

$$VH(\bar{P}) = \sum_{i=1}^{n} \frac{1-p_i}{n-1} \left(\ln\left(\frac{1-p_i}{n-1}\right)\right)^2 - \left(\sum_{i=1}^{n} \frac{1-p_i}{n-1} \ln\left(\frac{1-p_i}{n-1}\right)\right)^2.$$

To maximize this varentropy, we differentiate $VH(\bar{P})$ with respect to p_i and set the derivative equal to zero:

$$\frac{d}{dp_i} VH(\bar{P}) = 0.$$

Solving the system of equations results in the optimal solution:

$$p_i = \frac{1}{n}, \quad \forall i = 1, 2, \ldots, n.$$

Thus, the varentropy $VH(\bar{P})$ is maximized when $p_i = \frac{1}{n}$. Therefore, the maximum value of $VH(\bar{P})$ occurs when the original distribution is uniform.

$$VH(\bar{P}) = \text{manimized when } p_i = \frac{1}{n}.$$

□

Theorem 4.9. *Assume the $X = \{x_1, x_2, \ldots, x_n\}$, when the probability distribution satisfies $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$, the corresponding varextropy of the probability distribution after negation is minimised.*

Proof: The varextropy of a discrete probability distribution $P = \{p_1, p_2, \ldots, p_n\}$ is defined as:

$$VJ(P) = \sum_{i=1}^{n} (1-p_i) \left(\ln(1-p_i)\right)^2 - \left(\sum_{i=1}^{n} (1-p_i) \ln(1-p_i)\right)^2.$$

The negation of the probability distribution P, denoted as $\bar{P} = \{\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n\}$, is given by:

$$\bar{p}_i = \frac{1-p_i}{n-1}.$$

Thus, the negated varextropy $VJ(\bar{P})$ is:

$$VJ(\bar{P}) = \sum_{i=1}^{n} (1-\bar{p}_i) \left(\ln(1-\bar{p}_i)\right)^2 - \left(\sum_{i=1}^{n} (1-\bar{p}_i) \ln(1-\bar{p}_i)\right)^2.$$

Substituting $\bar{p}_i = \frac{1-p_i}{n-1}$ into this formula:

$$VJ(\bar{P}) = \sum_{i=1}^{n} \left(1 - \frac{1-p_i}{n-1}\right) \left(\ln\left(1 - \frac{1-p_i}{n-1}\right)\right)^2 - \left(\sum_{i=1}^{n} \left(1 - \frac{1-p_i}{n-1}\right) \ln\left(1 - \frac{1-p_i}{n-1}\right)\right)^2.$$

To maximize this varextropy, we differentiate $VJ(\bar{P})$ with respect to p_i and set the derivative equal to zero:

$$\frac{d}{dp_i} VJ(\bar{P}) = 0.$$

Solving the resulting system of equations leads to the optimal solution:

$$p_i = \frac{1}{n}, \quad \forall i = 1, 2, \dots, n.$$

Thus, the varextropy $VJ(\bar{P})$ is maximized when $p_i = \frac{1}{n}$. Therefore, the maximum value of $VJ(\bar{P})$ occurs when the original distribution is uniform.

$$VJ(\bar{P}) = \text{maximized when } p_i = \frac{1}{n}$$

□

In the following theorem, we show that repeated application of Yager's negation operator drives any probability distribution toward the uniform distribution.

Theorem 4.10. *Let $P^{(0)} = (p_1^{(0)}, \dots, p_n^{(0)})$ be any probability distribution on $X = \{x_1, \dots, x_n\}$ with $n \geq 3$. Define recursively $p_i^{(k+1)} = \frac{1-p_i^{(k)}}{n-1}$, $\quad i = 1, \dots, n$. Then the sequence $\{P^{(k)}\}$ converges to the uniform distribution $U = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$.*

Proof: Define the transformation

$$T(p_i) = \frac{1-p_i}{n-1}.$$

Rewrite it as

$$T(p_i) = \frac{1}{n-1} - \frac{1}{n-1} p_i.$$

Let

$$a = \frac{1}{n-1}.$$

Then

$$T(p_i) = a - a p_i.$$

Now observe that the uniform distribution U satisfies

$$T\left(\frac{1}{n}\right) = \frac{1 - \frac{1}{n}}{n-1} = \frac{1}{n}.$$

Thus, U is a fixed point.

Define the deviation from uniformity:

$$d_i^{(k)} = p_i^{(k)} - \frac{1}{n}.$$

Applying T,

$$d_i^{(k+1)} = p_i^{(k+1)} - \frac{1}{n} = \frac{1 - p_i^{(k)}}{n-1} - \frac{1}{n}.$$

After algebraic simplification,

$$d_i^{(k+1)} = -\frac{1}{n-1} d_i^{(k)}.$$

Thus,

$$d_i^{(k)} = \left(-\frac{1}{n-1}\right)^k d_i^{(0)}.$$

Since for $n \geq 3$,

$$\left|-\frac{1}{n-1}\right| < 1,$$

It follows that

$$\lim_{k\to\infty} d_i^{(k)} = 0.$$

Hence,

$$\lim_{k\to\infty} p_i^{(k)} = \frac{1}{n},$$

and therefore

$$\lim_{k\to\infty} P^{(k)} = U.$$

For $n = 2$, the factor equals -1, so the sequence oscillates between two distributions and does not converge unless the initial distribution is already uniform. □

5. CONCLUSION

In this paper, we developed an uncertainty-based framework for analysing the negation of discrete probability distributions under Yager's negation operator. Unlike earlier studies that primarily focused on entropy or extropy alone, our work integrates both first-order measures (Shannon entropy and extropy) and second-order measures (varentropy and varextropy), thereby providing a more comprehensive understanding of how negation transforms uncertainty.

We established that negation acts as an uncertainty-regularising transformation. For non-uniform probability distributions with $n \geq 3$, Shannon entropy and extropy increase under negation, while varentropy decreases toward zero and varextropy increases toward a stable limit. These results reveal that negation simultaneously increases the average level of uncertainty and reduces fluctuations in log-probabilities, driving the distribution toward structural balance. The uniform distribution plays a central role in this framework: it is invariant under negation and extremizes the considered measures. Moreover, we proved that repeated application of Yager's negation operator produces a convergent sequence that approaches the uniform distribution for $n \geq 3$, whereas in the binary case the process exhibits invariance without convergence dynamics.

The analysis demonstrates that probability negation is not merely a complementary transformation but a systematic mechanism for redistributing information in a way that enhances overall uncertainty while stabilizing variability. By extending the study of negation to second-order information measures, this work deepens the theoretical foundation of complementary uncertainty modelling.

Future research may explore generalized or nonlinear negation operators, continuous probability distributions, relationships with divergence measures and majorization theory, and applications in evidence theory, reliability analysis, machine learning, and decision-making under incomplete or adversarial information. The present study provides a theoretical basis for such developments and contributes to a broader understanding of complementary representations of uncertainty.

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