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ON THE ADMISSIBILITY OF ESTIMATORS OF TWO ORDERED GAMMA SCALE PARAMETERS UNDER ENTROPY LOSS FUNCTION

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Abstract:

- Suppose that a random sample of size n_i is drawn from a gamma distribution with known shape parameter $\nu_i > 0$ and unknown scale parameter $\beta_i > 0$, $i = 1, 2$, satisfying $0 < \beta_1 \leq \beta_2$. In estimation of β_1 and β_2 under the entropy loss function, we consider the class of mixed estimators of β_1 and β_2. It is shown that a subclass of mixed estimators of β_i beats the usual estimators $\overline{X_i}/\nu_i$, $i = 1, 2$, and the inadmissible estimators in the class of mixed estimators are derived. Also the asymptotic efficiency of mixed estimators relative to the usual estimators are obtained. Finally the results are extended to a subclass of the scale parameter exponential family and the family of transformed chi-square distributions.

Key-Words:

- *admissibility; entropy loss function; exponential family; gamma distribution; mixed estimators; ordered parameters.*

AMS Subject Classification:

- 62F30, 62C15, 62F10.

1. INTRODUCTION

When an ordering among parameters is known in advance, the problem of estimating the smallest or the largest parameters arises in various practical problems. For example, in estimating the mean lives of two components in which one is produced by a standard factory and the other is produced by a local factory, it is quite natural to assume an ordering among mean lives of the components that produced by two factory.

Estimating the ordered parameters has been considered by several researchers. For a classified and extensively reviewed work in this area, see van Eeden (2006). Suppose that an estimator is admissible when no information on the ordering of parameters is given. Then a natural question of interest is: Does this estimator remain admissible when it is assumed that the parameters are ordered?

A few researchers address this question for some well known distributions under the Squared Error Loss (SEL) and scale-invariant SEL function. For example, Katz (1963) introduced mixed estimators for simultaneous estimation of two ordered binomial parameters and showed that they are better than the unrestricted Maximum Likelihood Estimators (MLEs). Kumar and Sharma (1988) considered mixed estimators for two ordered normal means and discussed the minimaxity and inadmissibility of them. In estimating the ordered scale parameters of two exponential distributions Kaur and Singh (1991), Vijaysree and Singh (1991,1993), Kumar and Kumar (1993,1995), and Misra and Singh (1994) considered componentwise or simultaneous estimation of the ordered means of two exponential distributions and discussed the admissibility and inadmissibility of mixed estimators based on the sample means and the restricted MLEs. In estimating the ordered scale parameters of two gamma distributions, Misra *et al.* (2002) derived a smooth estimator that improves upon the best scale equivariant estimators, Chang and Shinozaki (2002) considered estimation of linear functions of the ordered scale parameters and Meghnatisi and Nematollahi (2009) considered admissibility and inadmissibility of mixed estimators of the ordered scale parameters when the shape parameters are arbitrary and known, see also Self and Liang (1987).

Suppose that $X_{ij},\ j=1,2,...,n_i,\ i=1,2,$ be two independent random samples from gamma distribution with known shape parameter $\nu_i>0$ and unknown scale parameter $\beta_i>0$, $i=1,2$, with probability density function (pdf)

$$f_{X_{ij}}(x)=\frac{1}{\beta_i^{\nu_i}\,\Gamma(\nu_i)}\,x^{\nu_i-1}e^{-x/\beta_i}\,,\qquad x>0,\quad \nu_i>0,\quad \beta_i>0\,,\quad j=1,...,n_i,\quad i=1,2\;. \tag{1.1}$$

We assume that $0<\beta_1\le\beta_2$, and want to estimate β_1 and β_2 component-wise.

It is interesting to note that in the literature, estimating the ordered parameters are often considered under the SEL and scale-invariant SEL function which are symmetric about the parameter value and convex in estimator δ. In some estimation problems, over-estimation may be more serious than under-estimation. For example, in estimating the average life of the components of an aircraft, over-estimation is usually more serious than under-estimation. In such cases, the usual methods of estimation, which are based on symmetric loss function may be inappropriate. In this regard, Misra *et al.* (2004) used asymmetric LINEX loss function to estimate the ordered parameters of two normal populations. As an alternative to scale-invariant SEL, which is appropriate for estimating the scale parameters β_1 and β_2, consider the entropy loss function given by

$$L(\beta_i, \delta_i) = \frac{\delta_i}{\beta_i} - \ln \frac{\delta_i}{\beta_i} - 1 \ , \qquad i = 1, 2 \ , \tag{1.2}$$

which is also known as Stein's loss. This loss is convex in δ_i and not symmetric, also it penalizes heavily under-estimation. In estimating the ordered parameters under the entropy loss function, Parsian and Nematollahi (1995) discussed the admissibility of usual estimators of the ordered Poisson parameters and Chang and Shinozaki (2008) compared the linear function of maximum likelihood and unbiased estimators of ordered gamma scale parameters and its reciprocals. For a review of the literature in using entropy loss, see Parsian and Nematollahi (1996) and references cited therein. Under the loss (1.2), the best scale invariant and admissible estimator of β_i under the model (1.1) is $\delta_i = \sum_{j=1}^{n_i} X_{ij}/n_i\nu_i = \overline{X_i}/\nu_i$, $i = 1, 2$ (see Dey *et al.*, 1987 and Nematollahi, 1995), and it is also the MLE of β_i, $i = 1, 2$.

In this paper we consider the class of mixed estimators of β_1 and β_2 under the model (1.1) with the restriction $0 < \beta_1 \leq \beta_2$, and discuss the admissibility and inadmissibility of the usual and mixed estimators of β_1 and β_2 under the entropy loss (1.2). To this end, in Section 2, a subclass of mixed estimators of β_i that beats the usual estimators $\delta_i = \overline{X_i}/\nu_i$, $i = 1, 2$, is obtained and the inadmissible estimators in the class of mixed estimators are identified. In Section 3, the admissible estimators in the class of mixed estimators are considered. The asymptotic efficiency of mixed estimators relative to the usual estimators are given in Section 4. In Section 5, the results are extended to a subclass of the scale parameter exponential family and also the family of transformed chi-square distributions introduced by Rahman and Gupta (1993).

2. INADMISSIBILITY RESULTS

Let $X_{i1}, X_{i2}, ..., X_{in_i}$, $i = 1, 2$, be two independent random samples from Gamma (ν_i, β_i)-distribution, $i = 1, 2$, with pdf (1.1) where $0 < \beta_1 \leq \beta_2$ are unknown and ν_1, ν_2 are known positive real valued shape parameters. Let $\gamma_i = n_i\nu_i$

and $\delta_i = \sum_{j=1}^{n_i} X_{ij}/\gamma_i = \overline{X_i}/\nu_i,\ i=1,2$. Then δ_1 and δ_2 are the ML, best scale equivariant and admissible estimators of β_1 and β_2, respectively, when β_1 and β_2 are unrestricted. Consider the mixed estimators

$$\delta_{1\alpha} = \min\bigl(\delta_1, \alpha\delta_1 + (1-\alpha)\delta_2\bigr)\ , \qquad 0 \le \alpha < 1\ , \tag{2.1}$$

and

$$\delta_{2\alpha} = \max\bigl(\delta_2, \alpha\delta_2 + (1-\alpha)\delta_1\bigr)\ , \qquad 0 \le \alpha < 1\ , \tag{2.2}$$

of β_1 and β_2, respectively. When $\alpha = \alpha_1 = \frac{\gamma_1}{\gamma_1+\gamma_2}$, then $\delta_{1\alpha}$ is the MLE of β_1 and if $\alpha = \alpha_2 = \frac{\gamma_2}{\gamma_1+\gamma_2}$, then $\delta_{2\alpha}$ is the MLE of β_2 when $\beta_1 \le \beta_2$, see Robertson *et al.* (1988) and Chang and Shinozaki (2002) for more details.

In this section, we identify the values of α such that $\delta_{i\alpha}$ is inadmissible among the class of mixed estimators of β_i and $\delta_{i\alpha}$ dominates the usual estimator δ_i of β_i, $i=1,2$. Let $R(\boldsymbol{\beta}, \delta_{i\alpha}) = E\bigl[\frac{\delta_{i\alpha}}{\beta_i} - \ln\frac{\delta_{i\alpha}}{\beta_i} - 1\bigr]$ and $R(\boldsymbol{\beta}, \delta_i) = E\bigl[\frac{\delta_i}{\beta_i} - \ln\frac{\delta_i}{\beta_i} - 1\bigr]$ be the risk functions of $\delta_{i\alpha}$ and δ_i, $i=1,2$, respectively. Also, let $y_1 = \beta_2/\beta_1$, $y_2 = \beta_1/\beta_2$ and $z = \gamma_1 y_1/(\gamma_1 y_1 + \gamma_2)$. Since $0 < \beta_1 \le \beta_2$, we have $y_1 \ge 1$, $0 < y_2 \le 1$ and $0 < z < 1$.

Theorem 2.1. *With* $\alpha_1 = \frac{\gamma_1}{\gamma_1+\gamma_2}$, *under the entropy loss function (1.2), for* $\alpha \in (\alpha_1, 1)$, $\gamma_1 > 1$ *and* $0 < \beta_1 \le \beta_2$,

$$R(\boldsymbol{\beta}, \delta_{1\alpha_1}) < R(\boldsymbol{\beta}, \delta_{1\alpha}) < R(\boldsymbol{\beta}, \delta_1)\ .$$

Proof: Let $T_1 = \frac{\gamma_2\delta_2}{\gamma_1 y_1\delta_1 + \gamma_2\delta_2}$ and $T_2 = \frac{\gamma_1\delta_1}{\beta_1} + \frac{\gamma_2\delta_2}{\beta_2}$. Then $\delta_1 = \frac{\beta_1 T_2(1-T_1)}{\gamma_1}$, $\delta_2 = \frac{\beta_2 T_1 T_2}{\gamma_2}$ and T_1 and T_2 are statistically independent with $T_1 \sim Beta(\gamma_2, \gamma_1)$ and $T_2 \sim Gamma(\gamma_1+\gamma_2, 1)$. Let $\Delta_1 = R(\boldsymbol{\beta}, \delta_1) - R(\boldsymbol{\beta}, \delta_{1\alpha})$, then

$$\begin{aligned}
\Delta_1 &= E\Biggl[\biggl\{\frac{\delta_1}{\beta_1} - \ln\frac{\delta_1}{\beta_1} - \frac{\alpha\delta_1 + (1-\alpha)\delta_2}{\beta_1} \\
&\qquad + \ln\frac{\alpha\delta_1 + (1-\alpha)\delta_2}{\beta_1}\biggr\}\, I_{[0,\infty)}(\delta_1-\delta_2)\Biggr] \\
&= E\Biggl[\biggl\{\frac{(1-\alpha)(\delta_1-\delta_2)}{\beta_1} + \ln\biggl(\alpha + (1-\alpha)\frac{\delta_2}{\delta_1}\biggr)\biggr\}\, I_{[0,\infty)}(\delta_1-\delta_2)\Biggr] \\
&= E\Biggl[\biggl\{\frac{1-\alpha}{\gamma_1\gamma_2}\Bigl(\gamma_2 - (\gamma_1 y_1 + \gamma_2)\, T_1\Bigr) T_2 \\
&\qquad + \ln\biggl(\alpha + (1-\alpha)\,\frac{\gamma_1 y_1 T_1}{\gamma_2(1-T_1)}\biggr)\biggr\}\, I_{0,1-z]}(T_1)\Biggr] \\
&= E\Bigl[f_{1\alpha}(T_1)\, I_{[0,1-z]}(T_1)\Bigr]\ ,
\end{aligned} \tag{2.3}$$

where

$$f_{1\alpha}(x) = \frac{(1-\alpha)\,(\gamma_1+\gamma_2)}{\gamma_1\gamma_2}\Big(\gamma_2 - (\gamma_1 y_1 + \gamma_2)\,x\Big) + \ln\left(\frac{\alpha\gamma_2(1-x) + (1-\alpha)\,\gamma_1 y_1 x}{\gamma_2(1-x)}\right). \tag{2.4}$$

From (2.4) and the distribution of T_1, the expectation (2.3) exist whenever $\gamma_1 > 1$. Now using the fact that $\ln x \geq 1 - \frac{1}{x}$ for $x > 0$, we have

$$\begin{aligned} f_{1\alpha}(x) \;\geq\; & \frac{(1-\alpha)\Big(\gamma_2 - (\gamma_1 y_1 + \gamma_2)\,x\Big)}{\gamma_1\gamma_2\Big(\alpha\gamma_2\,(1-x) + (1-\alpha)\,\gamma_1 y_1 x\Big)} \\ & \times \left[x(\gamma_1+\gamma_2)\Big((1-\alpha)\,\gamma_1 y_1 - \alpha\gamma_2\Big) + \alpha\gamma_2(\gamma_1+\gamma_2) - \gamma_1\gamma_2\right] \\ = \; & \frac{1-\alpha}{\gamma_1\gamma_2\Big[\alpha\gamma_2(1-x) + (1-\alpha)\,\gamma_1 y_1 x\Big]}\; g_{1\alpha}(x)\;, \end{aligned} \tag{2.5}$$

where

$$g_{1\alpha}(x) \;=\; A_1(y_1,\alpha)\,x^2 + B_1(y_1,\alpha)\,x + C_1(y_1,\alpha)\;, \tag{2.6}$$

and

$$\begin{aligned} A_1(y_1,\alpha) \;&=\; (\gamma_1+\gamma_2)\,(\gamma_1 y_1 + \gamma_2)\Big(\alpha\gamma_2 - (1-\alpha)\,\gamma_1 y_1\Big)\,, \\ B_1(y_1,\alpha) \;&=\; \gamma_2\Big[(\gamma_1 y_1 + \gamma_2)\Big(\gamma_1 - \alpha(\gamma_1+\gamma_2)\Big) \\ &\qquad + (\gamma_1+\gamma_2)\Big((1-\alpha)\,\gamma_1 y_1 - \alpha\gamma_2\Big)\Big]\,, \\ C_1(y_1,\alpha) \;&=\; \gamma_2^2\Big[\alpha(\gamma_1+\gamma_2) - \gamma_1\Big]\;. \end{aligned} \tag{2.7}$$

Note that $C_1(y_1,\alpha) > 0$ for all $y_1 \geq 1$ and $\alpha > \alpha_1$. When $A_1(y_1,\alpha) \neq 0$, the quadratic form (2.6) has the roots

$$x_1 = 1 - z \qquad \text{and} \qquad x_2 = 1 - z + \frac{\gamma_1\gamma_2^2\,(y_1-1)}{A_1(y_1,\alpha)}\;.$$

If $A_1(y_1,\alpha) > 0$, then $x_1 = 1 - z$ is the smaller positive root and if $A_1(y_1,\alpha) < 0$ then $x_1 = 1 - z$ is the only positive root when $\alpha \in (\alpha_1, 1)$. For the case $A_1(y_1,\alpha) = 0$, $x_1 = 1 - z$ is the only root. So, from (2.5), $f_{1\alpha}(x) > 0$ for $x \in [0, 1-z]$, and hence $\Delta_1 > 0$ for all $0 < \beta_1 \leq \beta_2$ when $\alpha \in (\alpha_1, 1)$, i.e., $R(\boldsymbol{\beta}, \delta_{1\alpha}) < R(\boldsymbol{\beta}, \delta_1)$ for all $\alpha \in (\alpha_1, 1)$ when $\gamma_1 > 1$.

Now from (2.3) and (2.4), when $\gamma_1 > 1$ we have

$$
\begin{aligned}
\frac{\partial R(\boldsymbol{\beta}, \delta_{1\alpha})}{\partial_\alpha} &= -\frac{\partial \Delta_1}{\partial \alpha} \\
&= E\left[\left\{\frac{\gamma_1+\gamma_2}{\gamma_1\gamma_2}\Big(\gamma_2 - (\gamma_1 y_1 + \gamma_2) T_1\Big)\right.\right. \\
&\qquad \left.\left. - \frac{\gamma_2(1-T_1) - \gamma_1 y_1 T_1}{\alpha\gamma_2(1-T_1) + (1-\alpha)\gamma_1 y_1 T_1}\right\} \times I_{[0,1-z]}(T_1)\right] \\
&= E\left[\frac{g_{1\alpha}(T_1)}{\gamma_1\gamma_2\Big\{\alpha\gamma_2(1-T_1) + (1-\alpha)\gamma_1 y_1 T_1\Big\}}\, I_{[0,1-z]}(T_1)\right],
\end{aligned}
\tag{2.8}
$$

where $g_{1\alpha}(x)$ is given by (2.6). For $\alpha \in (\alpha_1, 1)$ the above expectation is exist, and using a similar argument after relation (2.7), we conclude that $g_{1\alpha}(x) > 0$ for all $\alpha \in (\alpha_1, 1)$ and $x \in [0, 1-z]$. Therefore, from (2.8), $R(\boldsymbol{\beta}, \delta_{1\alpha})$ is an increasing function of α for $\alpha \in (\alpha_1, 1)$, i.e., $R(\boldsymbol{\beta}, \delta_{1\alpha_1}) < R(\boldsymbol{\beta}, \delta_{1\alpha})$ for all $\alpha \in (\alpha_1, 1)$ and $\gamma_1 > 1$, which completes the proof. □

To compare the risks of $\delta_{1\alpha_1}, \delta_{1\alpha}$ and δ_1, we use a Monte Carlo simulation study. First note that $\frac{\gamma_i\delta_i}{\beta_i} \sim Gamma(\gamma_i, 1)$, $i = 1, 2$, so the risk function of δ_i, $i = 1, 2$, under the entropy loss function (1.2) is given by

$$
\begin{aligned}
R(\beta_i, \delta_i) &= E\left[\frac{\delta_i}{\beta_i} - \ln\frac{\delta_i}{\beta_i} - 1\right] = 1 - E\left[\ln\frac{\gamma_i\delta_i}{\beta_i}\right] + \ln\gamma_i - 1 \\
&= -\frac{\Gamma'(\gamma_i)}{\Gamma(\gamma_i)} + \ln\gamma_i = \ln\gamma_i - \psi(\gamma_i), \qquad i = 1, 2,
\end{aligned}
\tag{2.9}
$$

where $\psi(\gamma_i) = \frac{\Gamma'(\gamma_i)}{\Gamma(\gamma_i)}$ is the digamma function. Using similar argument as in proof of Theorem 2.1, we have

$$
\begin{aligned}
R(\boldsymbol{\beta}, \delta_{1\alpha}) &= E\left[\frac{\delta_{1\alpha}}{\beta_1} - \ln\frac{\delta_{1\alpha}}{\beta_1} - 1\right] \\
&= E\left[\left(\frac{\alpha\delta_1 + (1-\alpha)\delta_2}{\beta_1} - \ln\frac{\alpha\delta_1 + (1-\alpha)\delta_2}{\beta_1} - 1\right) I_{[0,\infty)}(\delta_1 - \delta_2)\right. \\
&\qquad \left. + \left(\frac{\delta_1}{\beta_1} - \ln\frac{\delta_1}{\beta_1} - 1\right) I_{(0,\infty)}(\delta_2 - \delta_1)\right] \\
&= E\left[\left(\frac{\delta_1 - (1-\alpha)(\delta_1 - \delta_2)}{\beta_1} - \ln\left(\frac{\delta_1 - (1-\alpha)(\delta_1-\delta_2)}{\beta_1}\right) - 1\right)\right. \\
&\qquad \left. \times I_{[0,\infty)}(\delta_1 - \delta_2) + \left(\frac{\delta_1}{\beta_1} - \ln\frac{\delta_1}{\beta_1} - 1\right) I_{(0,\infty)}(\delta_2 - \delta_1)\right] \\
&= E\left[\left\{\left[\frac{T_2(1-T_1)}{\gamma_1} - \frac{1-\alpha}{\gamma_1\gamma_2}\left(\gamma_2 - \left(\frac{\gamma_1}{y_2} + \gamma_2\right)T_1\right)T_2\right]\right.\right. \\
&\qquad \left. - \ln\left[\frac{T_2(1-T_1)}{\gamma_1} - \frac{1-\alpha}{\gamma_1\gamma_2}\left(\gamma_2 - \left(\frac{\gamma_1}{y_2} + \gamma_2\right)T_1\right)T_2\right] - 1\right\} I_{(0,1-z]}(T_1) \\
&\qquad \left. + \left\{\frac{T_2(1-T_1)}{\gamma_1} - \ln\left(\frac{T_2(1-T_1)}{\gamma_1}\right) - 1\right\} I_{(1-z,1)}(T_1)\right].
\end{aligned}
\tag{2.10}
$$

Similarly $R(\boldsymbol{\beta}, \delta_{1\alpha_1})$ is obtained with replacing α by α_1 in (2.10). To calculate $R(\boldsymbol{\beta}, \delta_{1\alpha})$ in (2.10), we generate a random sample of size $m_1 = 1000$ from $T_1 \sim Beta(\gamma_2, \gamma_1)$ and a random sample of size $m_2 = 1000$ from $T_2 \sim Gamma(\gamma_1 + \gamma_2, 1)$ for some values of γ_1 and γ_2. Then by using Monte Carlo integration, the estimated risk of (2.10) is computed for α and α_1. Tables 1 and 2 show the risk of δ_1 and estimated risks of $\delta_{1\alpha_1}$ and $\delta_{1\alpha}$ for some values of γ_1, γ_2 and α. From these tables we observe that $R(\boldsymbol{\beta}, \delta_{1\alpha_1}) < R(\boldsymbol{\beta}, \delta_{1\alpha}) < R(\boldsymbol{\beta}, \delta_1)$ for $\alpha \in (\alpha_1, 1)$, which is proved analytically in Theorem 2.1.

Table 1: Estimated risks of $\delta_{1\alpha_1}$ and $\delta_{1\alpha}$ when $\gamma_1 = 1$ in comparison of $R(\boldsymbol{\beta}, \delta_1) = 0.5772$.

y_2	$\gamma_2 = 1, \quad \alpha = 0.6$		$\gamma_2 = 2, \quad \alpha = 0.5$		$\gamma_2 = 3, \quad \alpha = 0.4$	
	$R(\boldsymbol{\beta}, \delta_{1\alpha_1})$	$R(\boldsymbol{\beta}, \delta_{1\alpha})$	$R(\boldsymbol{\beta}, \delta_{1\alpha_1})$	$R(\boldsymbol{\beta}, \delta_{1\alpha})$	$R(\boldsymbol{\beta}, \delta_{1\alpha_1})$	$R(\boldsymbol{\beta}, \delta_{1\alpha})$
0.1	0.5481	0.5502	0.5474	0.5496	0.5256	0.5263
0.2	0.5419	0.5459	0.5209	0.5285	0.5273	0.5310
0.3	0.5515	0.5559	0.5261	0.5341	0.5012	0.5096
0.4	0.5643	0.5688	0.5087	0.5202	0.5103	0.5181
0.5	0.5080	0.5137	0.5306	0.5405	0.5050	0.5149
0.6	0.5192	0.5236	0.5100	0.5222	0.4724	0.4839
0.7	0.5051	0.5111	0.5424	0.5535	0.4652	0.4762
0.8	0.5430	0.5474	0.5258	0.5347	0.4743	0.4841
0.9	0.5341	0.5382	0.4586	0.4675	0.4603	0.4699
1.0	0.5123	0.5161	0.4914	0.4990	0.4581	0.4656

Table 2: Estimated risks of $\delta_{1\alpha_1}$ and $\delta_{1\alpha}$ when $\gamma_1 = 2$ in comparison of $R(\boldsymbol{\beta}, \delta_1) = 0.2704$.

y_2	$\gamma_2 = 2, \quad \alpha = 0.7$		$\gamma_2 = 3, \quad \alpha = 0.6$		$\gamma_2 = 4, \quad \alpha = 0.5$	
	$R(\boldsymbol{\beta}, \delta_{1\alpha_1})$	$R(\boldsymbol{\beta}, \delta_{1\alpha})$	$R(\boldsymbol{\beta}, \delta_{1\alpha_1})$	$R(\boldsymbol{\beta}, \delta_{1\alpha})$	$R(\boldsymbol{\beta}, \delta_{1\alpha_1})$	$R(\boldsymbol{\beta}, \delta_{1\alpha})$
0.1	0.2674	0.2685	0.2582	0.2589	0.2666	0.2668
0.2	0.2596	0.2619	0.2578	0.2602	0.2498	0.2514
0.3	0.2497	0.2542	0.2633	0.2674	0.2502	0.2538
0.4	0.2297	0.2369	0.2629	0.2685	0.2637	0.2679
0.5	0.2358	0.2433	0.2410	0.2485	0.2431	0.2500
0.6	0.2391	0.2468	0.2103	0.2194	0.2254	0.2317
0.7	0.2358	0.2451	0.2389	0.2481	0.2288	0.2371
0.8	0.2510	0.2589	0.2243	0.2338	0.2093	0.2155
0.9	0.2531	0.2618	0.2284	0.2352	0.2344	0.2392
1.0	0.2332	0.2395	0.2262	0.2338	0.2369	0.2412

Theorem 2.2. *With* $\alpha_2 = \frac{\gamma_2}{\gamma_1+\gamma_2} = 1-\alpha_1$, *under the entropy loss function (1.2), for* $\alpha \in (\alpha_2, 1)$, $\gamma_2 > 1$ *and* $0 < \beta_1 \le \beta_2$,

$$R(\boldsymbol{\beta}, \delta_{2\alpha_2}) < R(\boldsymbol{\beta}, \delta_{2\alpha}) < R(\boldsymbol{\beta}, \delta_2) \ .$$

Proof: Let $\Delta_2 = R(\boldsymbol{\beta}, \delta_2) - R(\boldsymbol{\beta}, \delta_{2\alpha})$, then using similar argument as in the proof of Theorem 2.1, we have

$$\begin{aligned}
\Delta_2 \;&=\; E\left[\left\{\frac{(1-\alpha)(\delta_2-\delta_1)}{\beta_2} + \ln\left(\alpha + (1-\alpha)\frac{\delta_1}{\delta_2}\right)\right\} I_{[0,\infty)}(\delta_1-\delta_2)\right] \\
&=\; E\left[\left\{\frac{1-\alpha}{\gamma_1\gamma_2}\Big((\gamma_1+\gamma_2 y_2)T_1 - \gamma_2 y_2\Big)T_2\right.\right. \\
&\qquad \left.\left. + \ln\left(\alpha + (1-\alpha)\frac{\gamma_2 y_2(1-T_1)}{\gamma_1 T_1}\right)\right\} I_{[0,1-z]}(T_1)\right] \qquad (2.11) \\
&=\; E\Big[f_{2\alpha}(T_1)\, I_{[0,1-z]}(T_1)\Big] \ ,
\end{aligned}$$

where

$$\begin{aligned}
f_{2\alpha}(x) \;&=\; \frac{(1-\alpha)(\gamma_1+\gamma_2)}{\gamma_1\gamma_2}\Big((\gamma_1+\gamma_2 y_2)x - \gamma_2 y_2\Big) \\
&\qquad + \ln\left(\frac{\alpha\gamma_1 x + (1-\alpha)\,\gamma_2 y_2(1-x)}{\gamma_1 x}\right) . \qquad (2.12)
\end{aligned}$$

From (2.12) and the distribution of T_1, the expectation (2.11) exists whenever $\gamma_2 > 1$. Now from (2.12) and the inequality $\ln(x) \ge 1 - \frac{1}{x}$ for $x > 0$, we have

$$f_{2\alpha}(x) \;\ge\; \frac{1-\alpha}{\gamma_1\gamma_2\Big[\alpha\gamma_1 x + (1-\alpha)\,\gamma_2 y_2(1-x)\Big]}\, g_{2\alpha}(x) \ , \qquad (2.13)$$

where

$$g_{2\alpha}(x) \;=\; A_2(y_2,\alpha)x^2 + B_2(y_2,\alpha)x + C_2(y_2,\alpha) \ , \qquad (2.14)$$

and

$$\begin{aligned}
A_2(y_2,\alpha) \;&=\; (\gamma_1+\gamma_2)\,(\gamma_1+\gamma_2 y_2)\Big(\alpha\gamma_1 - (1-\alpha)\gamma_2 y_2\Big) , \\
B_2(y_2,\alpha) \;&=\; \gamma_2\Big[(\gamma_1+\gamma_2 y_2)\Big((1-\alpha)\,(\gamma_1+\gamma_2)\,y_2 - \gamma_1\Big) \\
&\qquad - (\gamma_1+\gamma_2)\,y_2\Big(\alpha\gamma_1 - (1-\alpha)\gamma_2 y_2\Big)\Big] , \qquad (2.15) \\
C_2(y_2,\alpha) \;&=\; \gamma_2^2\, y_2\Big[\gamma_1 - (1-\alpha)\,(\gamma_1+\gamma_2)\,y_2\Big] \ .
\end{aligned}$$

Note that $C_2(y_2,\alpha) > 0$ and $A_2(y_2,\alpha) > 0$ for all $y_2 \le 1$ and $\alpha > \alpha_2$. The quadratic form (2.14) has the roots

$$x_1 = 1 - z \qquad \text{and} \qquad x_2 = 1 - z + \frac{\gamma_1^2\gamma_2(1-y_2)}{A_2(y_2,\alpha)} \ ,$$

and hence $x_1 = 1 - z$ is the smallest positive root. Hence, from (2.13), $f_{2\alpha}(x) > 0$ for $x \in [0, 1-z]$, and $\Delta_2 > 0$ for all $0 < \beta_1 \le \beta_2$ when $\alpha \in (\alpha_2, 1)$, which is shown that $R(\boldsymbol{\beta}, \delta_{2\alpha}) < R(\boldsymbol{\beta}, \delta_2)$ for all $\alpha \in (\alpha_2, 1)$ when $\gamma_2 > 1$.

Now, similar to the proof of Theorem 2.1, it is easy to show that for $\gamma_2 > 1$,

$$\begin{aligned} \frac{\partial R(\boldsymbol{\beta},\delta_{2\alpha})}{\partial\alpha} &= -\frac{\partial\Delta_2}{\partial\alpha} \\ &= E\Bigg[\Bigg\{\frac{\gamma_1+\gamma_2}{\gamma_1\gamma_2}\Big((\gamma_1+\gamma_2y_2)T_1-\gamma_2y_2\Big) \\ &\qquad - \frac{\gamma_1T_1-\gamma_2y_2(1-T_1)}{\alpha\gamma_1T_1+(1-\alpha)\gamma_2y_2(1-T_1)}\Bigg\}\times I_{[0,1-z]}(T_1)\Bigg] \\ &= E\left[\frac{g_{2\alpha}(T_1)}{\gamma_1\gamma_2\Big\{\alpha\gamma_1T_1+(1-\alpha)\gamma_2y_2(1-T_1)\Big\}}\ I_{[0,1-z]}(T_1)\right], \end{aligned} \tag{2.16}$$

where $g_{2\alpha}(x)$ is given by (2.14). Since $g_{2\alpha}(x) > 0$ for all $x \in [0, 1-z]$ and $\alpha \in (\alpha_2, 1)$, so from (2.16) $R(\boldsymbol{\beta}, \delta_{2\alpha})$ is an increasing function of α for $\alpha \in (\alpha_2, 1)$, i.e., $R(\boldsymbol{\beta}, \delta_{2\alpha_2}) < R(\boldsymbol{\beta}, \delta_{2\alpha})$ for all $\alpha \in (\alpha_2, 1)$ and $\gamma_2 > 1$, which completes the proof. □

Now we compare the risks of $\delta_{2\alpha_2}, \delta_{2\alpha}$ and δ_2. Similar to (2.10), we can show that

$$\begin{aligned} R(\boldsymbol{\beta},\delta_{2\alpha}) &= E\left[\frac{\delta_{2\alpha}}{\beta_2}-\ln\frac{\delta_{2\alpha}}{\beta_2}-1\right] \\ &= E\Bigg[\Bigg\{\left[\frac{T_1T_2}{\gamma_2}-\frac{1-\alpha}{\gamma_1\gamma_2}\Big((\gamma_1+\gamma_2y_2)T_1-\gamma_2y_2\Big)T_2\right] \\ &\qquad - \ln\left[\frac{T_1T_2}{\gamma_2}-\frac{1-\alpha}{\gamma_1\gamma_2}\Big((\gamma_1+\gamma_2y_2)T_1-\gamma_2y_2\Big)T_2\right]-1\Bigg\} \\ &\qquad \times I_{(0,1-z]}(T_1)+\left\{\frac{T_1T_2}{\gamma_2}-\ln\left(\frac{T_1T_2}{\gamma_2}\right)-1\right\}I_{(1-z,1)}(T_1)\Bigg]. \end{aligned} \tag{2.17}$$

To calculate $R(\boldsymbol{\beta}, \delta_{2\alpha})$ in (2.17), we use a Monte Carlo simulation study similar to the one used for computing (2.10). Tables 3 and 4 show the risk of δ_2 and estimated risks of $\delta_{2\alpha_2}$ and $\delta_{2\alpha}$ for some values of γ_1, γ_2 and α. From these tables we see that $R(\boldsymbol{\beta}, \delta_{2\alpha_2}) < R(\boldsymbol{\beta}, \delta_{2\alpha}) < R(\boldsymbol{\beta}, \delta_2)$ for $\alpha \in (\alpha_2, 1)$, which is proved analytically in Theorem 2.2.

Table 3: Estimated risks of $\delta_{2\alpha_2}$ and $\delta_{2\alpha}$ when $\gamma_2 = 2$ in comparison of $R(\boldsymbol{\beta}, \delta_2) = 0.2704$.

y_2	$\gamma_1 = 1, \quad \alpha = 0.8$		$\gamma_1 = 2, \quad \alpha = 0.7$		$\gamma_1 = 3, \quad \alpha = 0.6$	
	$R(\boldsymbol{\beta}, \delta_{2\alpha_2})$	$R(\boldsymbol{\beta}, \delta_{2\alpha})$	$R(\boldsymbol{\beta}, \delta_{2\alpha_2})$	$R(\boldsymbol{\beta}, \delta_{2\alpha})$	$R(\boldsymbol{\beta}, \delta_{2\alpha_2})$	$R(\boldsymbol{\beta}, \delta_{2\alpha})$
0.1	0.2452	0.2469	0.2610	0.2634	0.2531	0.2538
0.2	0.2551	0.2629	0.2454	0.2499	0.2544	0.2610
0.3	0.2468	0.2557	0.2256	0.2354	0.2273	0.2377
0.4	0.2248	0.2378	0.2062	0.2204	0.2039	0.2184
0.5	0.2189	0.2303	0.1947	0.2126	0.1985	0.2141
0.6	0.2045	0.2185	0.1838	0.2025	0.1787	0.1962
0.7	0.2151	0.2281	0.1846	0.2009	0.1644	0.1796
0.8	0.2149	0.2272	0.1784	0.1938	0.1569	0.1718
0.9	0.1912	0.2017	0.1849	0.1968	0.1568	0.1672
1.0	0.1942	0.2002	0.1607	0.1723	0.1444	0.1539

Table 4: Estimated risks of $\delta_{2\alpha_2}$ and $\delta_{2\alpha}$ when $\gamma_2 = 3$ in comparison of $R(\boldsymbol{\beta}, \delta_2) = 0.1758$.

y_2	$\gamma_1 = 2, \quad \alpha = 0.7$		$\gamma_1 = 3, \quad \alpha = 0.6$		$\gamma_1 = 4, \quad \alpha = 0.5$	
	$R(\boldsymbol{\beta}, \delta_{2\alpha_2})$	$R(\boldsymbol{\beta}, \delta_{2\alpha})$	$R(\boldsymbol{\beta}, \delta_{2\alpha_2})$	$R(\boldsymbol{\beta}, \delta_{2\alpha})$	$R(\boldsymbol{\beta}, \delta_{2\alpha_2})$	$R(\boldsymbol{\beta}, \delta_{2\alpha})$
0.1	0.1706	0.1711	0.1655	0.1658	0.1689	0.1692
0.2	0.1719	0.1735	0.1644	0.1656	0.1610	0.1618
0.3	0.1567	0.1595	0.1628	0.1649	0.1623	0.1638
0.4	0.1568	0.1608	0.1458	0.1494	0.1527	0.1552
0.5	0.1419	0.1474	0.1325	0.1370	0.1362	0.1392
0.6	0.1340	0.1402	0.1371	0.1415	0.1276	0.1309
0.7	0.1359	0.1419	0.1174	0.1226	0.1155	0.1188
0.8	0.1281	0.1335	0.1208	0.1252	0.1031	0.1065
0.9	0.1194	0.1242	0.1159	0.1184	0.1024	0.1041
1.0	0.1220	0.1242	0.1035	0.1055	0.1007	0.1015

Remark 2.1. Theorem 2.1 shows that for $\alpha \in (\alpha_1, 1)$ the mixed estimators (2.1) are inadmissible and are beaten by the MLE $\delta_{1\alpha_1}$ of β_1 when $\gamma_1 > 1$. Also Theorem 2.2 show that for $\alpha \in (\alpha_2, 1)$ the mixed estimators (2.2) are inadmissible and are beaten by the MLE $\delta_{2\alpha_2}$ of β_2 when $\gamma_2 > 1$. If $\gamma_1 = \gamma_2 = \gamma$, i.e., $n_1\nu_1 = n_2\nu_2$, then $\alpha_1 = \alpha_2 = \frac{1}{2}$ and the mixed estimators $\delta_{1\alpha}$ and $\delta_{2\alpha}$ are inadmissible for $\alpha \in (\frac{1}{2}, 1)$ when $\gamma > 1$. Note that this is the case when $n_1 = n_2$ and $\nu_1 = \nu_2$.

3. ADMISSIBILITY RESULTS

In this section, for the case $\gamma_1 = \gamma_2 = \gamma$ and $\gamma > 1$, we discuss the admissibility of $\delta_{1\alpha}$ and $\delta_{2\alpha}$ for β_1 and β_2 in the class of mixed estimators (2.1) and (2.2), respectively. As noted in Remark 2.1, these estimators are inadmissible when $\alpha \in (\frac{1}{2}, 1)$. So, we discuss their admissibility for $\alpha \in [0, \frac{1}{2}]$ in the sequel.

(i) Admissibility of $\delta_{2\alpha}$

For deriving admissible estimators in the class of mixed estimators (2.2), we find values of α that minimizes the risk function $R(\boldsymbol{\beta}, \delta_{2\alpha})$. From (2.16) with $\gamma_1 = \gamma_2 = \gamma$ and $\gamma > 1$, we have

$$\begin{aligned}\frac{\partial R(\boldsymbol{\beta}, \delta_{2\alpha})}{\partial \alpha} &= E\Bigg[\Bigg\{2\Big((1+y_2)T_1 - y_2\Big) \\ &\qquad - \frac{(1+y_2)T_1 - y_2}{\alpha\big\{(1+y_2)T_1 - y_2\big\} + y_2(1-T_1)}\Bigg\}\, I_{[0,\frac{y_2}{1+y_2}]}(T_1)\Bigg]\end{aligned} \tag{3.1}$$

which is a strictly increasing function of α, i.e., $R(\boldsymbol{\beta}, \delta_{2\alpha})$ for fixed $\boldsymbol{\beta}$ is a strictly convex function of α. Therefore for $\alpha > 0$, $\gamma > 1$ and fixed $\boldsymbol{\beta}$, $R(\boldsymbol{\beta}, \delta_{2\alpha})$ will be minimized at the point α given by $\frac{\partial R(\boldsymbol{\beta},\delta_{2\alpha})}{\partial \alpha} = 0$ which reduces to

$$\begin{aligned}&E\Bigg[\Bigg\{\frac{2}{y_1} - \frac{1}{\alpha_2(y_1,\gamma)\,\big\{(1+y_1)T_1 - 1\big\} + (1-T_1)}\Bigg\} \\ &\qquad\qquad \times \Big\{(1+y_1)T_1 - 1\Big\}\, I_{[0,\frac{1}{1+y_1}]}(T_1)\Bigg] = 0\,.\end{aligned} \tag{3.2}$$

For $y_1 = 1$, (3.2) reduces to

$$\Big(2\,\alpha_2(1,\gamma) - 1\Big)\, E\left[\frac{(2\,T_1 - 1)^2}{\alpha_2(1,\gamma)\,\{2\,T_1 - 1\} + (1-T_1)}\, I_{[0,\frac{1}{2}]}(T_1)\right] = 0\,. \tag{3.3}$$

Since the expectation in (3.3) is finite for $\alpha_2(1,\gamma) > 0$ and $\gamma > 1$, so (3.3) has the root $\alpha_2(1,\gamma) = \frac{1}{2}$. From (3.2), $\alpha_2(y_1,\gamma)$ is a continuous function of $y_1 \geq 1$ but the behavior of $\alpha_2(y_1,\gamma)$ can not be determined analytically. The graph of $\alpha_2(y_1,\gamma)$ as a function of $y_1 \geq 1$ for different values of $\gamma > 1$ are shown in Figure 1. From this figure we observe that $\alpha_2(y_1,\gamma)$ decreases as y_1 or γ or both increases, and for fixed γ, $\alpha_2(y_1,\gamma) \to -\infty$ as $y_1 \to \infty$. Therefore for each $\alpha \in [0, \frac{1}{2}]$ there is a y_1 for which $R(\boldsymbol{\beta}, \delta_{2\alpha})$ is minimum, which implies that for $\alpha \in [0, \frac{1}{2}]$, $\delta_{2\alpha}$ is admissible in the class of mixed estimators. So, we have the following conjecture.

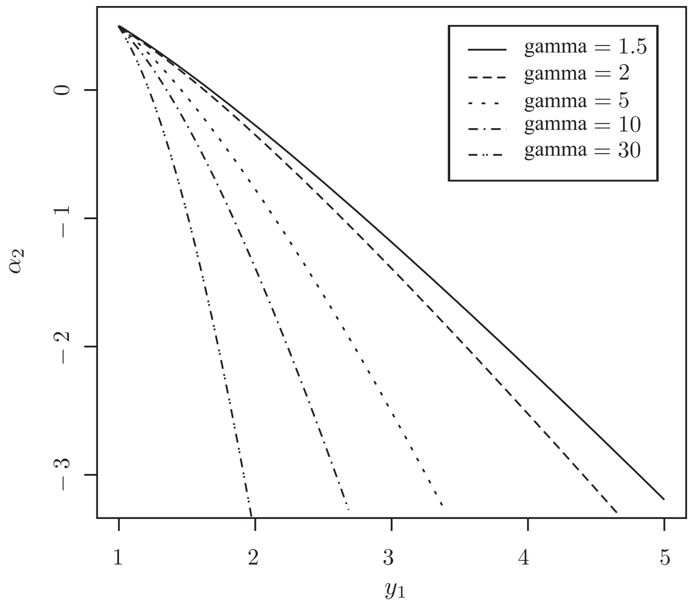

Figure 1: Graph of $\alpha_2(y_1, \gamma)$ for different values of γ.

Conjecture 3.1. *For $\gamma_1 = \gamma_2 = \gamma$ and $\gamma > 1$, under the entropy loss function (1.2), the estimator $\delta_{2\alpha}$ in the class of mixed estimators (2.2) is admissible if and only if $\alpha \in [0, \frac{1}{2}]$.*

Remark 3.1. From (3.1) we have

$$\begin{aligned} \frac{\partial R(\boldsymbol{\beta}, \delta_{2\alpha})}{\partial \alpha} &= E\Bigg[\left\{\frac{2}{y_1} - \frac{1}{\alpha_2(y_1,\gamma)\big\{(1+y_1)T_1 - 1\big\} + (1-T_1)}\right\} \\ &\qquad \times \Big\{(1+y_1)T_1 - 1\Big\}\ I_{[0,\frac{1}{1+y_1}]}(T_1)\Bigg], \end{aligned}$$

and for $y_1 > 2$,

$$\frac{2}{y_1} < 1 < \frac{1}{1-T_1} < \frac{1}{\alpha_2(y_1,\gamma)\big\{(1+y_1)T_1 - 1\big\} + (1-T_1)},$$

so, $\frac{\partial R(\boldsymbol{\beta},\delta_{2\alpha})}{\partial \alpha} > 0$ when $y_1 > 2$. Therefore the minimum value $\alpha_2(y_1,\gamma)$ of $R(\boldsymbol{\beta},\delta_{2\alpha})$ is attained when $1 \le y_1 < 2$, so we only need the graph of $\alpha_2(y_1,\gamma)$ for $1 \le y_1 < 2$ (see Figure 1).

(ii) Admissibility of $\delta_{1\alpha}$

Similarly, From (2.8) with $\gamma_1 = \gamma_2 = \gamma$ and $\gamma > 1$, we have

$$\frac{\partial R(\boldsymbol{\beta}, \delta_{1\alpha})}{\partial \alpha} = E\left[\left\{2\Big(1-(1+y_1)T_1\Big) - \frac{1-(1+y_1)T_1}{\alpha\big\{1-(1+y_1)T_1\big\}+y_1T_1}\right\} I_{[0,\frac{1}{1+y_1}]}(T_1)\right],$$

which is a strictly increasing function of α, i.e., $R(\boldsymbol{\beta}, \delta_{1\alpha})$ for fixed $\boldsymbol{\beta}$ is a strictly convex function of α. Therefore, for $\alpha > 0$, $\gamma > 1$ and fixed $\boldsymbol{\beta}$, $R(\boldsymbol{\beta}, \delta_{1\alpha})$ will be minimized at the point α given by $\frac{\partial R(\boldsymbol{\beta},\delta_{1\alpha})}{\partial \alpha} = 0$ which reduces to

$$E\left[\left\{2 - \frac{1}{\alpha_1(y_1,\gamma)\,\big\{1-(1+y_1)T_1\big\}+y_1T_1}\right\} \times \Big\{1-(1+y_1)T_1\Big\}\, I_{[0,\frac{1}{1+y_1}]}(T_1)\right] = 0\,. \tag{3.4}$$

Similar to part (i), for $y_1 = 1$, (3.4) has the root $\alpha_1(1,\gamma) = \frac{1}{2}$. From (3.4), $\alpha_1(y_1,\gamma)$ is a continuous function of $y_1 \geq 1$ but the behavior of $\alpha_1(y_1,\gamma)$ can not be determined analytically. The graph of $\alpha_1(y_1,\gamma)$ as a function of $y_1 \geq 1$ for different values of $\gamma > 1$ are shown in Figure 2. From this figure we can not determine the minimum value of α for each $\gamma > 1$. So, the admissibility or inadmissibility of $\delta_{1\alpha}$ for $\alpha \in [0, \frac{1}{2})$ remain unsolved.

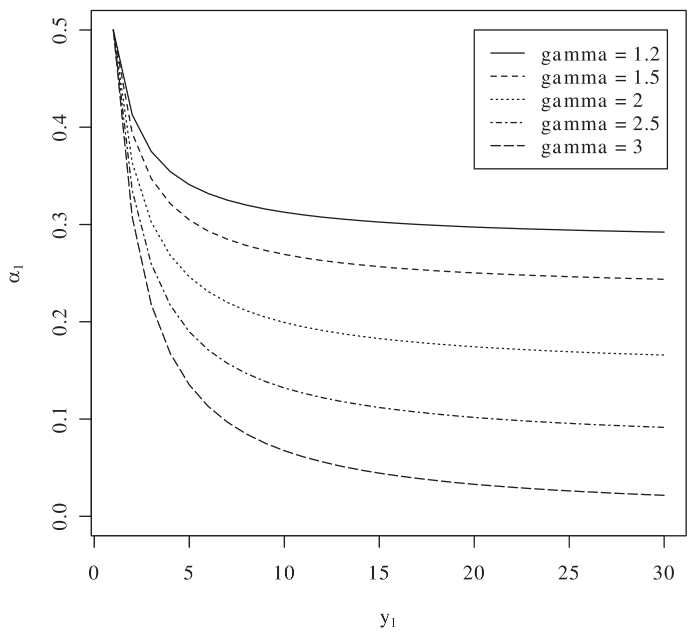

Figure 2: Graph of $\alpha_1(y_1,\gamma)$ for different values of γ.

Remark 3.2. The above argument shows that for $y_1 = 1$, $R(\boldsymbol{\beta}, \delta_{1\alpha})$ and $R(\boldsymbol{\beta}, \delta_{2\alpha})$ minimized at $\alpha_1(1,\gamma) = \frac{1}{2}$ and $\alpha_2(1,\gamma) = \frac{1}{2}$, respectively. So, for $\gamma_1 = \gamma_2 = \gamma$ and $\gamma > 1$, the MLEs $\delta_{1,\frac{1}{2}}$, $\delta_{2,\frac{1}{2}}$ are admissible for β_1 and β_2 among the class of mixed estimators (2.1) and (2.2), respectively.

4. EFFICIENCY OF MIXED ESTIMATORS

Let $e(\delta_{i\alpha}, \delta_i) = R(\boldsymbol{\beta}, \delta_i)/R(\boldsymbol{\beta}, \delta_{i\alpha})$ denote the efficiency of $\delta_{i\alpha}$ relative to δ_i, $i = 1, 2$. In Section 2, we derived conditions for which $\delta_{i\alpha}$, $i = 1, 2$, is more efficient than δ_i, $i = 1, 2$. Since $R(\boldsymbol{\beta}, \delta_i)$ and $R(\boldsymbol{\beta}, \delta_{i\alpha})$ are positive, so $e(\delta_{i\alpha}, \delta_i) > 0$ for $i = 1, 2$. In this section, we compare the asymptotic efficiency of these mixed estimators relative to usual estimators.

From (2.9), we have $R(\boldsymbol{\beta}, \delta_i) = \ln\gamma_i - \psi(\gamma_i)$, $i = 1, 2$. Note that for $\gamma_i > 0$, $\frac{1}{2\gamma_i} < \ln\gamma_i - \psi(\gamma_i) < \frac{1}{\gamma_i}$, $i = 1, 2$.

Theorem 4.1. *Let $\gamma_1 = \gamma_2 = \gamma$ and $\gamma > 1$, then for $0 \le \alpha < 1$ and for $i = 1, 2$,*

(**a**) $\lim_{y_1\to\infty} e(\delta_{i\alpha}, \delta_i) = 1$ *for all* $\gamma > 1$.

(**b**) $\lim_{\gamma\to\infty} e(\delta_{i\alpha}, \delta_i) = 1$ *for all* $0 < \beta_1 < \beta_2$.

Proof: (**a**) For $i = 1$, from (2.3) and (2.9) with $\gamma_1 = \gamma_2 = \gamma$ and $\gamma > 1$ we have

$$\left|1 - \frac{R(\boldsymbol{\beta}, \delta_{1\alpha})}{R(\boldsymbol{\beta}, \delta_1)}\right| = \frac{1}{\ln\gamma - \psi(\gamma)} \left|E[f_{1\alpha}(T_1)]\, I_{[0,\frac{1}{1+y_1}]}(T_1)\right| \le A(\gamma, y_1) \int_0^{z_1} |f_{1\alpha}(x)|\, dx$$

where $A(\gamma, y_1) = \frac{\Gamma(2\gamma)\,(z_1(1-z_1))^{\gamma-1}}{\Gamma^2(\gamma)\,[\ln\gamma - \psi(\gamma)]}$, $z_1 = \frac{1}{1+y_1}$ and $f_{1\alpha}(x)$ is given by (2.4). Notice that

$$\begin{aligned} |f_{1\alpha}(x)| &= \left|2(1-\alpha)\Big(1 - (1+y_1)x\Big) + \ln\frac{\alpha(1-x) + (1-\alpha)y_1 x}{1-x}\right| \\ &\le 2(1-\alpha)\Big[1 - (1+y_1)x\Big] - \ln\frac{\alpha(1-x) + (1-\alpha)y_1 x}{1-x}\,. \end{aligned}$$

Now, if $\alpha = 0$ then $|f_{1\alpha}(x)| \le 2[1 - (1+y_1)x] - \ln\frac{x}{1-x} - \ln y_1$ and

$$\begin{aligned} \left|1 - \frac{R(\boldsymbol{\beta}, \delta_{1\alpha})}{R(\boldsymbol{\beta}, \delta_1)}\right| &\le A(\gamma, y_1)\left\{\frac{-\big[1 - (1+y_1)x\big]^2}{1+y_1} - x\ln x \right. \\ &\qquad \left. - (1-x)\ln(1-x) - x\ln y_1 \Big|_0^{\frac{1}{1+y_1}}\right\} \qquad (4.1) \\ &= A(\gamma, y_1)\, B_1(y_1)\,, \end{aligned}$$

where

$$B_1(y_1) \;=\; \frac{1}{1+y_1} + \ln\left(\frac{1+y_1}{y_1}\right). \tag{4.2}$$

If $0<\alpha<1$, then using the fact $\ln x \geq 1-\frac{1}{x}$, $x>0$, we have

$$|f_{1\alpha}(x)| \;\leq\; 2(1-\alpha)\Big[1-x(1+y_1)\Big] + \frac{(1-\alpha)\left[1-(1+y_1)x\right]}{\alpha(1-x)+(1-\alpha)y_1 x}\ ,$$

and

$$\begin{aligned}\left|1-\frac{R(\boldsymbol{\beta},\delta_{1\alpha})}{R(\boldsymbol{\beta},\delta_1)}\right| \;&=\; A(\gamma,y_1)\left\{\frac{-(1-\alpha)\left[1-(1+y_1)x\right]}{1+y_1} - \left[\frac{(1-\alpha)\,(1+y_1)}{y_1-\alpha(1+y_1)}\right]\right.\\ &\qquad \times\left[x-\frac{\alpha\ln\big(\alpha(1-x)+(1-\alpha)y_1x\big)}{y_1-\alpha(1+y_1)}\right.\\ &\qquad \left.\left.-\frac{\ln\big(\alpha(1-x)+(1-\alpha)y_1x\big)}{1+y_1}\right]\right\}\Bigg|_0^{\frac{1}{1+y_1}}\\ &=\; A(\gamma,y_1)\,B_2(\alpha,y_1)\ ,\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}B_2(\alpha,y_1) \;&=\; (1-\alpha)\left[\frac{1}{1+y_1}-\frac{1}{y_1-\alpha(1+y_1)}\right.\\ &\qquad\left.\times\left\{1-\frac{y_1}{y_1-\alpha(1+y_1)}\,\ln\left(\frac{y_1}{\alpha(1+y_1)}\right)\right\}\right].\end{aligned} \tag{4.4}$$

It is easy to verify that when $\alpha\in(0,1)$, $B_1(y_1)\to 0$ and $B_2(\alpha,y_1)\to 0$ as $y_1\to\infty$. Also $0\leq A(\gamma,y_1)\leq\frac{\Gamma(2\gamma)\,(\frac{1}{4})^\gamma}{\Gamma^2(\gamma)\,[\ln\gamma-\psi(\gamma)]}$. So from (4.1) and (4.3), $\lim\limits_{y_1\to\infty}\left|1-\frac{R(\boldsymbol{\beta},\delta_{1\alpha})}{R(\boldsymbol{\beta},\delta_1)}\right|=0$ for all $\alpha\in[0,1)$, i.e., $\lim\limits_{y_1\to\infty} e(\delta_{1\alpha},\delta_1)=1$ for all $\alpha\in[0,1)$ and $\gamma>1$, which completes the proof for $i=1$. For $i=2$, the proof is similar.

(**b**) For $0<\beta_1<\beta_2$ (i.e., $0<z_1<1$) we have

$$0\;\leq\;A(\gamma,y_1)\;\leq\;\frac{2\,\gamma\,\Gamma(2\gamma)}{\Gamma^2(\gamma)}\left(\frac{y_1}{(1+y_1)^2}\right)^{\gamma-1}=\;\frac{\gamma^2\,\Gamma(2\gamma+1)}{\Gamma^2(\gamma+1)}\Big(z_1(1-z_1)\Big)^{\gamma-1}.$$

Using Stirling's approximation formula $\big(\Gamma(\gamma+1)\simeq\gamma^{\gamma+\frac{1}{2}}\,e^{-\gamma}\sqrt{2\pi}\,\big)$, we have

$$0\;\leq\;A(\gamma,y_1)\;\leq\;\frac{4}{\sqrt{2\pi}}\,\gamma^{\frac{3}{2}}\Big(4\,z_1(1-z_1)\Big)^{\gamma-1}$$

which tends to zero as $\gamma\to\infty$. Now from (4.1)–(4.4), $\lim\limits_{\gamma\to\infty}\left|1-\frac{R(\boldsymbol{\beta},\delta_{1\alpha})}{R(\boldsymbol{\beta},\delta_1)}\right|=0$, i.e., $\lim\limits_{\gamma\to\infty} e(\delta_{i\alpha},\delta_i)=1$ for all $0<\beta_1<\beta_2$ and $\alpha\in[0,1)$, which completes the proof for $i=1$. For $i=2$, the proof is similar. □

5. EXTENSION TO A SUBCLASS OF EXPONENTIAL FAMILY

Let $\mathbf{X}_i = (X_{i1}, X_{i2}, ..., X_{in_i})$, $i = 1, 2$, has the joint probability density function

$$f(\mathbf{x}_i, \theta_i) = C(\mathbf{x}_i, n_i)\, \theta_i^{-m_i}\, e^{-T_i(\mathbf{x}_i)/\theta_i} , \qquad i = 1, 2 , \tag{5.1}$$

where $\mathbf{x}_i = (x_{i1}, ..., x_{in_i})$, $C(\mathbf{x}_i, n_i)$ is a function of $\mathbf{x}_i$ and n_i, $\theta_i = \tau_i^r$ for some $r > 0$, m_i is a function of n_i and $T_i(\mathbf{x}_i)$ is a complete sufficient statistic for θ_i with $Gamma(m_i, \theta_i)$-distribution. For example, $Exponential(\beta_i)$ with $\theta_i = \beta_i$, $Gamma(\nu_i, \beta_i)$ with $\theta_i = \beta_i$ and known v_i, *Inverse Gaussian*(∞, λ_i) with $\theta_i = \frac{1}{\lambda_i}$, $Normal(0, \sigma_i^2)$ with $\theta_i = \sigma_i^2$, $Weibull(\eta_i, \beta_i)$ with $\theta_i = \eta_i^{\beta_i}$ and known β_i, $Rayleigh(\beta_i)$ with $\theta_i = \beta_i^2$, *Generalized Gamma*$(\alpha_i, \lambda_i, p_i)$ with $\theta_i = \lambda_i^{p_i}$ and known p_i and α_i, *Generalized Laplace*(λ_i, k_i) with $\theta_i = \lambda_i^{k_i}$ and known k_i belong to the family of distributions (5.1). An admissible linear estimator of $\theta_i = \tau_i^r$ in this family under the entropy loss function can be found in Parsian and Nematollahi (1996).

Since $T_i = T_i(\mathbf{X}_i)$, $i = 1, 2$, has a $Gamma(m_i, \theta_i)$-distribution, therefore we can extend the results of Sections 2–4 to the subclass of exponential family (5.1) by replacing $\gamma_i = n_i\nu_i$, β_i and $\sum_{j=1}^{n_i} X_{ij} = \gamma_i\delta_i$ by m_i, θ_i and $T_i(\mathbf{X}_i)$, respectively.

The results of Sections 2–4 can be extended to some other families of distributions which do not necessarily belong to a scale families, such as Pareto or beta distributions. A family of distributions that includes these distributions as special cases, is the family of transformed chi-square distributions which is originally introduced by Rahman and Gupta (1993). They considered the one parameter exponential family

$$f(\mathbf{x}_i, \eta_i) = e^{a_i(\mathbf{x}_i)\, b(\eta_i) + c(\eta_i) + h(\mathbf{x}_i)} , \qquad i = 1, 2 , \tag{5.2}$$

and showed that $-2\, a_i(\mathbf{X}_i)\, b(\eta_i)$ has a $Gamma\big(\frac{k_i}{2}, 2\big)$-distribution if and only if

$$\frac{2\, c'(\eta_i)\, b(\eta_i)}{b'(\eta_i)} = k_i . \tag{5.3}$$

When k_i is an integer, $-2\, a_i(\mathbf{X}_i)\, b(\eta_i)$ follow a chi-square distribution with k_i degrees of freedom. They called the one parameter exponential family (5.2) which satisfies (5.3), the family of transformed chi-square distributions. For example, beta, Pareto, exponential, lognormal and some other distributions belong to this family of distributions (see Table 1 of Rahman and Gupta,1993).

Now it is easy to show that if condition (5.3) holds then the one parameter exponential family (5.2) is in the form of the scale parameter exponential family (5.1) with $m_i = \frac{k_i}{2}$, $T_i(\mathbf{X}_i) = a_i(\mathbf{X}_i)$ and $\theta_i = -1/b(\eta_i)$ (see Jafari Jozani *et al.*, 2002). Hence with these substitutions, we can extend the results of Sections 2–4 to the family of transformed chi-square distributions.

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