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Fractional-Order Generalized Inaccuracy Measures Based on Copula and Associated Properties and Applications

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Abstract:

- In this work, we propose some inaccuracy measures based on fractional-order generalized survival and distributional copula functions. Several associated properties, including bounds of the inaccuracy measures, are obtained. Jeffreys' inaccuracy measures, using the survival and distribution copula functions-based inaccuracy measures, are also introduced and are studied. Moreover, estimation of the proposed measures using a semiparametric technique is discussed further. Finally, a trivariate real dataset relating to the "Pima Indian Diabetes" is considered, and a model selection is made on it through the proposed inaccuracy measure for the purpose of illustration.

Keywords:

- *inaccuracy measure; copula; Jeffreys' inaccuracy; semiparametric estimation; mean squared error; model selection.*

AMS Subject Classification:

- 94A17; 60E15; 62B10.

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1. INTRODUCTION

An inaccuracy measure was proposed by Kerridge (1961) as a generalization of Shannon's uncertainty measure. The inaccuracy measure is useful to compute the difference between two statistical distributions. It mainly quantifies how much one distribution deviates from another distribution. Let $f(\cdot)$ be the actual probability density function (pdf) of a non-negative absolutely continuous random variable X and $g(\cdot)$ be its estimated pdf. Then, the Kerridge inaccuracy measure is given by

$$K(X||Y) = \int_0^\infty f(x)(-\ln g(x))dx = E_X(-\ln g(X)), \tag{1.1}$$

provided the integral exists. Sometimes, it is interpreted as the average uncertainty involved in the incorrect assumption by the experimenter. The inaccuracy measure reduces to the Shannon entropy when both the distributions coincide, that is, $g(\cdot) = f(\cdot)$. Due to the importance of the inaccuracy measure, many authors have studied its properties. For example, Taneja et al. (2009) explored the inaccuracy measure between two residual lifetime distributions. Kumar et al. (2011) studied inaccuracy measure for past lifetime distributions. The cumulative versions of the inaccuracy measure for the residual as well as past lifetime distributions have been proposed by Kundu et al. (2016). Moreover, the cumulative residual and past inaccuracy measures have been extended to bivariate set-up by Ghosh and Kundu (2019).

There have been growing interest in studying information measures by making use of fractional calculus. Ubriaco (2009) proposed the entropy measure based on fractional calculus, and showed that the fractional entropy satisfies all the properties of Shannon's entropy except additivity. Using the properties of fractional calculus, Machado (2014) proposed a novel expression for entropy. Karci (2016) presented a new method for fractional order derivative to probability distribution functions, and then obtained two new definitions for entropy computation. Subsequently, many other authors considered fractional order information or divergence measures. In this direction, one may refer to the works of Mao et al. (2020), Di Crescenzo et al. (2021), Dong et al. (2022), Saha and Kayal (2023), Khan et al. (2023) Kumbhakar and Tsai (2023), Kharazmi and Contreras-Reyes (2024), and Saha and Kayal (2025). However, there has been a recent interest to study information measures by using copula functions although it started in 2011 when Ma and Sun (2011) proved that the mutual information of random variables is equal to the negative copula entropy. Ghosh and Sunoj (2024) investigated some properties of the copula-based mutual information, while Saha and Kayal (2026) proposed copula-based extropy measures and discussed several properties of them.

Prompted by these developments, we propose here multivariate fractional-order generalized copula inaccuracy measures (CIMs) based on both survival and distributional copula functions, which generalize some of the existing copula-based measures. Some properties of the survival copula-based multivariate fractional-order generalized CIMs are also established. Next, some semiparametric inferential methods are developed. Finally, the measure is shown to be useful in selecting a better model. To the best of our knowledge, no such method has been discussed in the literature so far.

The rest of this paper is structured as follows. In Section 2, we have proposed fractional-order generalized survival CIM and have discussed its properties. We have shown that many existing copula-based measures can be obtained from the proposed measure. Various bounds are also obtained. In Section 3, we have introduced the Jeffreys' inaccuracy measures and have discussed some of their properties. In Section 4, semiparametric approach is made use of estimating the proposed measures. Section 5 provides an application related to model selection criteria for a trivariate dataset relating to the Pima Indian Diabetes. Finally, Section 6 presents some brief concluding remarks.

2. FRACTIONAL-ORDER GENERALIZED SURVIVAL INACCURACY MEASURE BASED ON COPULA

In this section, we introduce and study fractional generalized inaccuracy measures based on the survival copula and distribution copula between two continuous random vectors $\mathbf{X} = (\mathbf{X_1}, \dots, \mathbf{X_n})$ and $\mathbf{Y} = (\mathbf{Y_1}, \dots, \mathbf{Y_n})$. First, we state the definition of an n-dimensional copula. See Nelsen (2006) and Durante and Sempi (2016) for details.

Definition 2.1. For every $n \geq 2$, an n-dimensional copula is an n-dimensional distribution function concentrated on $[0, 1]^n$ whose univariate marginals are uniformly distributed on $[0, 1]$.

We use $C(\mathbf{u}) = \mathbf{C}(\mathbf{u_1}, \dots, \mathbf{u_n})$ to denote a copula. It satisfies the following properties.

(i) $C(u_1, \dots, u_n)$ is non-decreasing in each of its components, u_i;

(ii) The i-th marginal distribution is obtained by setting $u_j = 1$ for $j \neq i$ and since it it is uniformly distributed, we have

$$C(1, \dots, 1, u_i, 1 \dots, 1) = u_i;$$

(iii) For $a_i \leq b_i$, $P(U_1 \in [a_1, b_1], \dots, U_n \in [a_n, b_n])$ must be non-negative. This implies the rectangular inequality

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(u_{1,i_1}, \dots, u_{n,i_n}) \geq 0,$$

where $u_{j,1} = a_j$ and $u_{j,2} = b_j$.

The inaccuracy measures are defined based on the survival copulas of $\mathbf{X}$ and $\mathbf{Y}$, denoted by $\widehat{C}_{\mathbf{X}}$ and $\widehat{C}_{\mathbf{Y}}$, respectively, and their corresponding distribution copulas, say $C_{\mathbf{X}}$ and $C_{\mathbf{Y}}$. We recall that the survival copula couples a joint survival function with the marginal SFs, while the distribution copula couples a joint cumulative distribution function (CDF) with the marginal distribution functions. Denote by $\overline{F}_{\mathbf{X}}$ the joint SF of $\mathbf{X}$, and $\overline{F}_i$ the marginal SFs of X_i's, for $i = 1, \dots, n$. Then, $\overline{F}_{\mathbf{X}}$ can be expressed as

$$\overline{F}_{\mathbf{X}}(x_1, \dots, x_n) = P(X_1 > x_1, \dots, X_n > x_n) = \widehat{C}_{\mathbf{X}}(\overline{F}_1(x_1), \dots, \overline{F}_n(x_n)). \tag{2.1}$$

The survival copula is appropriate, for example, for random variables associated with durations exceeding over threshold values. Similarly, the joint CDF of $\mathbf{X}$ can be represented in terms of the distribution copula function as

$$F_{\mathbf{X}}(x_1,\ldots,x_n) = P(X_1 \le x_1,\ldots,X_n \le x_n) = C_{\mathbf{X}}(F_1(x_1),\ldots,F_n(x_n)), \tag{2.2}$$

where F_i denotes the marginal CDF of X_i, for $i = 1,\ldots,n$. We now use (2.1) and (2.2) to define inaccuracy measures as follows.

Definition 2.2. The fractional-order generalized survival copula inaccuracy (FGSCI) measure of order α between $\mathbf{X}$ and $\mathbf{Y}$ is defined as

$$K_\alpha(\mathbf{X}||\mathbf{Y}) = \frac{\mathbf{1}}{\mathbf{\Gamma}(\alpha+\mathbf{1})}\int_{\mathbf{0}}^{\mathbf{1}}\cdots\int_{\mathbf{0}}^{\mathbf{1}} \widehat{\mathbf{C}}_{\mathbf{X}}(\mathbf{u})(-\ln \widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u}))^{\alpha}\mathbf{du}, \quad \alpha > \mathbf{0}, \tag{2.3}$$

where $\mathbf{u} = (\mathbf{u_1},\ldots,\mathbf{u_n})$, "ln" denotes the natural logarithm, and $\Gamma(\cdot)$ denotes the complete gamma function.

Definition 2.3. The fractional-order generalized copula inaccuracy (FGCI) measure of order α between $\mathbf{X}$ and $\mathbf{Y}$ is defined as

$$K^*_\alpha(\mathbf{X}||\mathbf{Y}) = \frac{\mathbf{1}}{\mathbf{\Gamma}(\alpha+\mathbf{1})}\int_{\mathbf{0}}^{\mathbf{1}}\cdots\int_{\mathbf{0}}^{\mathbf{1}} \mathbf{C}_{\mathbf{X}}(\mathbf{u})(-\ln \mathbf{C}_{\mathbf{Y}}(\mathbf{u}))^{\alpha}\mathbf{du}, \quad \alpha > \mathbf{0}. \tag{2.4}$$

The above proposed measures will be useful in measuring the error due to the wrong assignment of a survival copula function $\widehat{C}_{\mathbf{Y}}$ or a distribution copula function $C_{\mathbf{Y}}$ by an experimenter in place of the correct survival copula function $\widehat{C}_{\mathbf{X}}$ or the distribution copula function $C_{\mathbf{X}}$, respectively. Note that the above defined measures are always positive. It is of interest to point out the following observations from (2.3) and (2.4):

(*a*) For $\alpha = 1$, $K_\alpha(\mathbf{X}||\mathbf{Y})$ becomes survival CIM, whereas $K^*_\alpha(\mathbf{X}||\mathbf{Y})$ becomes distribution CIM;

(*b*) For $\widehat{C}_{\mathbf{X}}(\mathbf{u}) = \widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u})$, $K_\alpha(\mathbf{X}||\mathbf{Y})$ and $K^*_\alpha(\mathbf{X}||\mathbf{Y})$ reduce to the fractional generalized survival and distribution copula entropy measures, respectively;

(*c*) For $\widehat{C}_{\mathbf{X}}(\mathbf{u}) = \widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u})$ and $\alpha = 1$, $K_\alpha(\mathbf{X}||\mathbf{Y})$ and $K^*_\alpha(\mathbf{X}||\mathbf{Y})$ become survival copula and distribution copula entropy measures, respectively. For survival and distribution copula entropy measures in bivariate case, one may refer to Sunoj and Nair (2025) and Arshad et al. (2024), respectively.

Next, we present some examples to illustrate the derivation of these measures. For this purpose, we consider some specific bivariate survival copulas. Note that the following known relation can be used to obtain the survival copula from a copula:

$$\widehat{C}_{(X_1,X_2)}(u_1,u_2) \;=\; u_1+u_2-1+C_{(X_1,X_2)}(1-u_1,1-u_2). \tag{2.5}$$

Let us denote the lower incomplete gamma function by

$$\Gamma^*(a;\beta) = \int_0^a e^{-z}z^{\beta-1}dz. \tag{2.6}$$

Example 2.1. Consider the Fréchet-Hoeffding lower bound copula, $C_{(X_1,X_2)}(u_1,u_2) = \max\{u_1+u_2-1, 0\}$, which gives a lower bound for every bivariate copula. The corresponding survival copula is obtained from (2.5) to be

$$\widehat{C}_{(X_1,X_2)}(u_1,u_2) = u_1+u_2-1+\max\{1-u_1-u_2, 0\} \equiv \begin{cases} u_1+u_2-1 & \text{if } u_1+u_2 \geq 1, \\ 0 & \text{if } u_1+u_2 < 1. \end{cases} \tag{2.7}$$

Further, consider the product copula, $C_{(Y_1,Y_2)}(u_1,u_2) = u_1u_2$, for which the corresponding survival copula is

$$\widehat{C}_{(Y_1,Y_2)}(u_1,u_2) = u_1u_2. \tag{2.8}$$

Then, the FGSCI between (X_1,X_2) and (Y_1,Y_2) is obtained as

$$\begin{aligned}
&K_\alpha((X_1,X_2)||(Y_1,Y_2)) \\
&= \frac{1}{\Gamma(1+\alpha)}\int_0^1\int_0^1 (u_1+u_2-1+\max\{1-u_1-u_2,0\})(-\ln u_1u_2)^\alpha u_1 du_2 \\
&= \frac{1}{\Gamma(1+\alpha)}\iint\limits_{0<u_1,u_2<1,\ u_1+u_2\geq 1} (u_1+u_2-1)(-\ln u_1-\ln u_2)^\alpha\, du_1\, du_2 \\
&= \frac{1}{\Gamma(1+\alpha)}\int_{u_1=0}^1\int_{u_2=1-u_1}^1 (u_1+u_1-1)(-\ln u_1-\ln u_2)^\alpha du_2du_1 \\
&= \frac{1}{\Gamma(1+\alpha)}\int_{u_1=0}^1\int_{w=0}^{u_1} w(-\ln u_1-\ln(w-u_1+1))^\alpha dwdu_1 \ (Let\ w = u_2+u_1-1) \\
&= \frac{1}{\Gamma(1+\alpha)}\int_{u_1=0}^1\int_{v=0}^{1} u_1^2 v(-\ln u_1-\ln(1-u_1(1-v)))^\alpha dvdu_1 \ (Taking\ v = \frac{w}{u_1}) \\
&= \frac{1}{\Gamma(1+\alpha)}\int_{u_1=0}^1\int_{z=-\ln u_1}^{-\ln(u_1(1-u_1))} (e^{-z}-u_1(1-u_1))\frac{z^\alpha e^{-z}}{u_1^2} dzdu_1.
\end{aligned} \tag{2.9}$$

Now, using the transformation $z = -\ln u_1 - \ln(1-u_1(1-v))$, the FGSCI can be further simplified as

$$\begin{aligned}
&K_\alpha((X_1,X_2)||(Y_1,Y_2)) \\
&= \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\int_0^1 \frac{1}{u_1^2}\Big\{\Gamma^*(-2\ln(u_1(1-u_1));\alpha+1)-\Gamma^*(-2\ln u_1;\alpha+1)\Big\}du_1 \\
&\quad -\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(\frac{1}{u_1}-1\right)\Big\{\Gamma^*(-\ln(u_1(1-u_1));\alpha+1)-\Gamma^*(-\ln u_1;\alpha+1)\Big\}du_1,
\end{aligned} \tag{2.10}$$

where $\Gamma^*(\cdot;\cdot)$ is as given in (2.6). We have plotted this measure in Figure 1(a) for $\alpha > 0$. From Figure 1(a), we observe that the values of the FGSCI increases, as $\alpha > 0$ increases.

Example 2.2. Consider the Fréchet-Hoeffding upper bound copula given by

$$C_{(X_1,X_2)}(u_1,u_2) = \min\{u_1,u_2\},$$

providing an upper bound for every bivariate copula. The corresponding survival copula is obtained from (2.5) as

$$\widehat{C}_{(X_1,X_2)}(u_1,u_2) = u_1+u_2-1+\min\{1-u_1,1-u_2\} \equiv \begin{cases} u_2 \ if\ u_1 \geq u_2 \\ u_1 \ if\ u_1 < u_2. \end{cases} \tag{2.11}$$

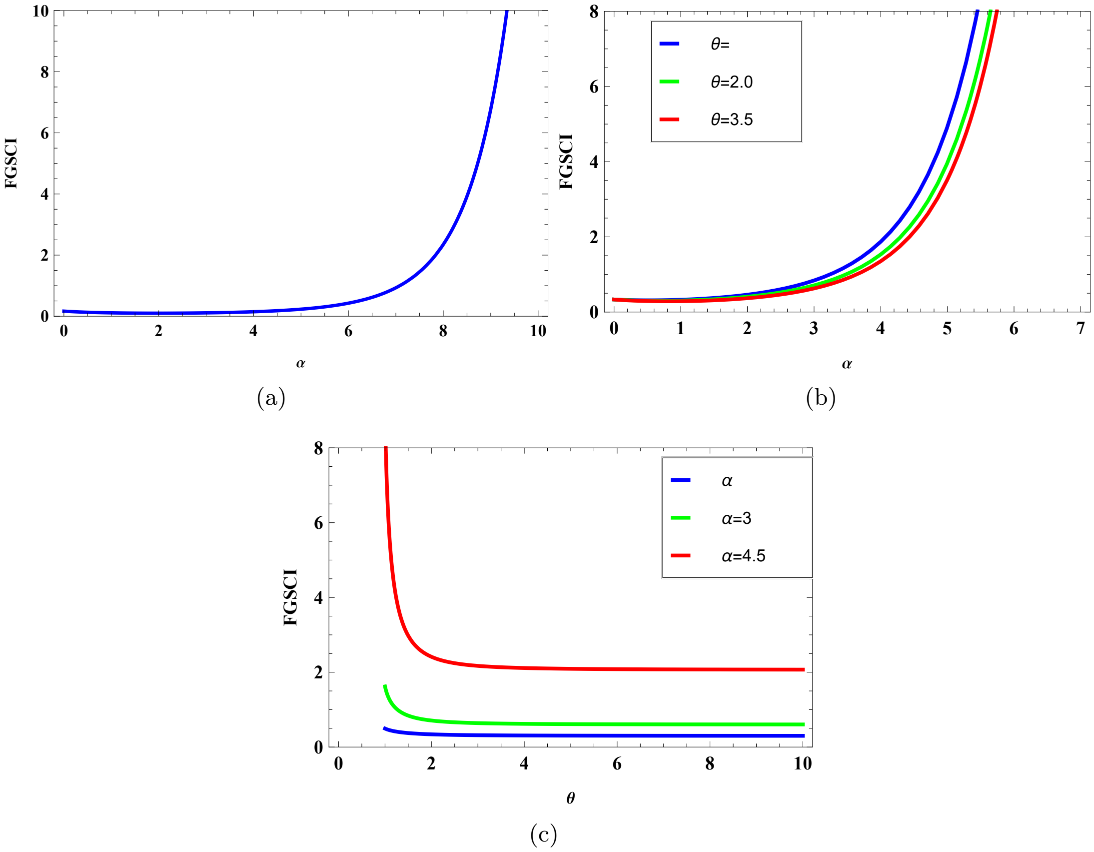

Figure 1: Plots of the FGSCI measure (a) for Fréchet-Hoeffding lower bound copula and product copula with respect to α in Example 2.1; (b) for Fréchet-Hoeffding upper bound copula and Gumbel copula with respect to α for $\theta = 1.5,\ 2.0$ and 3.5; and (c) for Fréchet-Hoeffding upper bound copula and Gumbel copula with respect to θ for $\alpha = 1.5,\ 3.0$ and 4.5 as in Example 2.2. The values of θ are chosen randomly.

Additionally, we consider Gumbel copula, which is asymmetric in nature and provides more weight in the right side tail, given by

$$C_{(Y_1,Y_2)}(u_1,u_2) = e^{-[(-\ln u_1)^\theta + (-\ln u_2)^\theta]^{\frac{1}{\theta}}}, \quad \theta \geq 1. \tag{2.12}$$

The Gumbel copula, known as logistic copula, is also an Archimedian copula. Further, it is known as the extreme value copula since $C_{(Y_1,Y_2)}(u_1^\beta, u_2^\beta) = C_{(Y_1,Y_2)}(u_1,u_2)^\beta$, for $\beta > 0$. The corresponding survival copula can be obtained as

$$\widehat{C}_{(Y_1,Y_2)}(u_1,u_2) = u_1 + u_2 - 1 + e^{-[(-\ln(1-u_1))^\theta + (-\ln(1-u_2))^\theta]^{\frac{1}{\theta}}}, \quad \theta \geq 1. \tag{2.13}$$

Then, the FGSCI between (X_1, X_2) and (Y_1, Y_2) is given by

$$
\begin{aligned}
&K_\alpha\big((X_1, X_2)||(Y_1, Y_2)\big)\\
&= \frac{1}{\Gamma(1+\alpha)} \int_0^1 \int_0^1 (u_1 + u_2 - 1 + \min\{1-u_1, 1-u_2\})\\
&\times \Big(-\ln(u_1 + u_2 - 1 + e^{-[(-\ln(1-u_1))^\theta + (-\ln(1-u_2))^\theta]^{\frac{1}{\theta}}})\Big)^\alpha du_1 du_2\\
&= \frac{1}{\Gamma(1+\alpha)} \int_0^1 \int_0^{u_1} u_2 \Big(-\ln(u_1 + u_2 - 1 + e^{-[(-\ln(1-u_1))^\theta + (-\ln(1-u_2))^\theta]^{\frac{1}{\theta}}})\Big)^\alpha du_2 du_1\\
&+ \frac{1}{\Gamma(1+\alpha)} \int_0^1 \int_{u_1}^1 u_1 \Big(-\ln(u_1 + u_2 - 1 + e^{-[(-\ln(1-u_1))^\theta + (-\ln(1-u_2))^\theta]^{\frac{1}{\theta}}})\Big)^\alpha du_2 du_1.
\end{aligned}
\tag{2.14}
$$

We have plotted (2.14) in Figure 1(b) with respect to $\alpha > 0$ for several choices of θ. Similarly, Figure 1(c) depicts plots of FGSCI with respect to θ for different choices of α. From Figures 1(b) and (c), it is clear that the FGSCI increases as α increases for all θ and the FGSCI decreases as θ increases for all $\alpha > 0$ and that is stable for $\theta > 2$ for the example.

Example 2.3. Consider the Cuadras-Augé (see Cuadras and Augé, 1981) and product copula functions given by

$$
C_{(X_1,X_2)}(u_1, u_2) = \begin{cases} u_1 u_2^{1-\theta} \text{ if } u_1 \le u_2 \\ u_1^{1-\theta} u_2 \text{ if } u_2 \le u_1 \end{cases}
\tag{2.15}
$$

and

$$
C_{(Y_1,Y_2)}(u_1, u_2) = u_1 u_2,
\tag{2.16}
$$

respectively, where $0 < \theta < 1$. Then, the FGCI measure of order α between (X_1, X_2) and (Y_1, Y_2) is obtained as

$$
\begin{aligned}
&K^*_\alpha\big((X_1, X_2)||(Y_1, Y_2)\big)\\
&= \frac{1}{\Gamma(1+\alpha)} \left[\int_0^1 \int_0^{u_2} u_1 u_2^{1-\theta} (-\ln u_1 u_2)^\alpha du_1 du_2 + \int_0^1 \int_{u_2}^1 u_1^{1-\theta} u_2 (-\ln u_1 u_2)^\alpha du_1 du_2 \right]\\
&= \frac{1}{\Gamma(1+\alpha)} [I_1 + I_2],
\end{aligned}
\tag{2.17}
$$

where

$$
\begin{aligned}
I_1 &= \int_0^1 \int_0^{u_2} u_1 u_2^{1-\theta} (-\ln u_1 u_2)^\alpha du_1 du_2\\
&= \int_0^1 \frac{1}{u_2^{\theta+1}} \int_{-\ln u_2^2}^\infty e^{-2t} t^\alpha dt du_2 \ (using\ t = -\ln u_1 u_2)\\
&= \int_0^\infty e^{-2t} t^\alpha \int_{e^{-\frac{t}{2}}}^1 \frac{1}{u_2^{\theta+1}} du_2 dt = \frac{\Gamma(\alpha+1)}{\theta(2-\frac{\theta}{2})^{\alpha+1}} - \frac{\Gamma(\alpha+1)}{\theta 2^{\alpha+1}}.
\end{aligned}
\tag{2.18}
$$

Moreover, we have

$$
I_2 = \int_0^1 \int_{u_2}^1 u_2 u_1^{1-\theta} (-\ln u_1 u_2)^\alpha du_1 du_2 = \frac{\Gamma(\alpha+1)}{\theta(2-\frac{\theta}{2})^{\alpha+1}} - \frac{\Gamma(\alpha+1)}{\theta 2^{\alpha+1}}.
\tag{2.19}
$$

Thus, using the values of I_1 and I_2 in (2.17), we finally obtain

$$
K^*_\alpha\big((X_1, X_2)||(Y_1, Y_2)\big) = \frac{1}{\theta} \left[\frac{2}{(2-\frac{\theta}{2})^{\alpha+1}} - \frac{1}{2^\alpha} \right].
\tag{2.20}
$$

Example 2.4. Consider the Cuadras-Augé copula $C_{(X_1,X_2)}(u_1,u_2)$ in Example 2.3 and minimum copula function

$$C_{(Y_1,Y_2)}(u_1,u_2)=\min\{u_1,u_2\}. \tag{2.21}$$

Then, the FGCI measure of order α between (X_1,X_2) and (Y_1,Y_2) is obtained as

$$\begin{aligned}&K^*_\alpha\big((X_1,X_2)||(Y_1,Y_2)\big)\\&=\frac{1}{\Gamma(1+\alpha)}\left[\int_0^1\int_0^{u_2}u_1u_2^{1-\theta}(-\ln u_1)^\alpha du_1du_2+\int_0^1\int_{u_2}^1 u_1^{1-\theta}u_2(-\ln u_2)^\alpha du_1du_2\right]\\&=\frac{1}{\Gamma(1+\alpha)}[I_{11}+I_{12}],\end{aligned} \tag{2.22}$$

where

$$I_{11}=I_{12}=\int_0^1 u_2^{1-\theta}\int_0^{u_2}u_1(-\ln u_1)^\alpha du_1du_2=\frac{\Gamma(\alpha+1)}{2-\theta}\left[\frac{1}{2^{\alpha+1}}-\frac{1}{(4-\theta)^{\alpha+1}}\right]. \tag{2.23}$$

Thus, we obtain

$$K^*_\alpha\big((X_1,X_2)||(Y_1,Y_2)\big)=\frac{2}{2-\theta}\left[\frac{1}{2^{\alpha+1}}-\frac{1}{(4-\theta)^{\alpha+1}}\right]. \tag{2.24}$$

Remark 2.1. It needs to be mentioned here that we have not presented similar results for the FGCI measure for the sake of brevity.

Below, we obtain bounds for the FSCI in terms of the survival copula-based inaccuracy measure.

Proposition 2.1. *For two random vectors* $\mathbf{X}$ *and* $\mathbf{Y}$*, we have*

$$K_\alpha(\mathbf{X}||\mathbf{Y})\begin{cases}\le \frac{1}{\Gamma(1+\alpha)}(K(\mathbf{X}||\mathbf{Y}))^\alpha & \text{if } 0<\alpha<1\\ \ge \frac{1}{\Gamma(1+\alpha)}(K(\mathbf{X}||\mathbf{Y}))^\alpha & \text{if } \alpha>1,\end{cases}$$

where $K(\mathbf{X}||\mathbf{Y})=-\int_0^1\cdots\int_0^1\widehat{C}_{\mathbf{X}}(\mathbf{u})\ln\widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u})d\mathbf{u}$ *is the generalized inaccuracy measure based on survival copula functions.*

Proof: From Definition 2.2, we have

$$\begin{aligned}K_\alpha(\mathbf{X}||\mathbf{Y})&=\frac{1}{\Gamma(1+\alpha)}\int_0^1\cdots\int_0^1\widehat{C}_{\mathbf{X}}(\mathbf{u})(-\ln\widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u}))^\alpha d\mathbf{u}\\&=\frac{1}{\Gamma(1+\alpha)}\int_0^1\cdots\int_0^1(-\widehat{C}^{\frac{1}{\alpha}}_{\mathbf{X}}(\mathbf{u})\ln\widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u}))^\alpha d\mathbf{u}.\end{aligned} \tag{2.25}$$

Moreover, $\psi(t)=t^\alpha$ is concave and convex with respect to $t>0$ when $0<\alpha<1$ and $\alpha>1$, respectively. Using this in (2.25), we obtain, for $0<\alpha<1$, that

$$\begin{aligned}K_\alpha(\mathbf{X}||\mathbf{Y})&\le\frac{1}{\Gamma(1+\alpha)}\left(-\int_0^1\cdots\int_0^1\widehat{C}^{\frac{1}{\alpha}}_{\mathbf{X}}(\mathbf{u})\ln\widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u})d\mathbf{u}\right)^\alpha\\&\le\frac{1}{\Gamma(1+\alpha)}\left(-\int_0^1\cdots\int_0^1\widehat{C}_{\mathbf{X}}(\mathbf{u})\ln\widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u})d\mathbf{u}\right)^\alpha\\&=\frac{1}{\Gamma(1+\alpha)}(K(\mathbf{X}||\mathbf{Y}))^\alpha,\end{aligned} \tag{2.26}$$

where the first inequality is due to Jensen's inequality of the integrals and the second inequality is due to the fact that $\widehat{C}_{\mathbf{X}}^{\frac{1}{\alpha}}(\mathbf{u}) \leq \widehat{\mathbf{C}}_{\mathbf{X}}(\mathbf{u})$ for $0 < \alpha < 1$. For $\alpha > 1$, the proof is similar. This completes the proof of the proposition. □

The concordance order (see Joe, 1990; Cebrián et al., 2004) is an important concept in practice. This order can be used to understand the level of concordance between the random variables, which are represented by copulas. A survival copula $\widehat{C}_1$ is said to be smaller than the survival copula $\widehat{C}_2$ if

$$\widehat{C}_1(u,v) \leq \widehat{C}_2(u,v), \text{ for all } (u,v) \in J^2, \tag{2.27}$$

where $J = [0,1]$, and is denoted by $\widehat{C}_1 \prec \widehat{C}_2$. Note that the positive dependence order and the concordance order are equivalent. Below, we obtain an inequality between inaccuracy measures using the concept of concordance order.

Proposition 2.2. *Consider three random vectors* $\mathbf{X}$, $\mathbf{Y}$ *and* $\mathbf{Z}$ *with respective survival copulas* $\widehat{C}_{\mathbf{X}}$, $\widehat{C}_{\mathbf{Y}}$ *and* $\widehat{C}_{\mathbf{Z}}$. *Further, let* $\widehat{C}_{\mathbf{Y}} \prec \widehat{C}_{\mathbf{Z}}$. *Then,*

$$K_\alpha(\mathbf{X}||\mathbf{Y}) \geq \mathbf{K}_\alpha(\mathbf{X}||\mathbf{Z}).$$

Proof: Under the assumption made, we have $\widehat{C}_{\mathbf{Y}}(\mathbf{u}) \leq \widehat{\mathbf{C}}_{\mathbf{Z}}(\mathbf{u})$, for all $\mathbf{u} \in \mathbf{J}^{\mathbf{n}}$. Then, for $\alpha > 0$, we obtain

$$\widehat{C}_{\mathbf{X}}(\mathbf{u})(-\ln \widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u}))^\alpha \geq \widehat{\mathbf{C}}_{\mathbf{X}}(\mathbf{u})(-\ln \widehat{\mathbf{C}}_{\mathbf{Z}}(\mathbf{u}))^\alpha. \tag{2.28}$$

Now, the required result follows upon integrating both sides of the inequality in (2.28). This completes the proof of the proposition. □

Proposition 2.3. *Consider three random vectors* $\mathbf{X}$, $\mathbf{Y}$ *and* $\mathbf{Z}$ *with respective survival copulas* $\widehat{C}_{\mathbf{X}}$, $\widehat{C}_{\mathbf{Y}}$ *and* $\widehat{C}_{\mathbf{Z}}$. *Further, let* $\widehat{C}_{\mathbf{X}} \prec \widehat{C}_{\mathbf{Y}}$. *Then,*

$$K_\alpha(\mathbf{X}||\mathbf{Z}) \leq \mathbf{K}_\alpha(\mathbf{Y}||\mathbf{Z}).$$

Proof: We have $\widehat{C}_{\mathbf{X}}(\mathbf{u}) \leq \widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u})$, for all $\mathbf{u} \in \mathbf{J}^{\mathbf{n}}$. Then, for $\alpha > 0$, we obtain

$$\widehat{C}_{\mathbf{X}}(\mathbf{u})(-\ln \widehat{\mathbf{C}}_{\mathbf{Z}}(\mathbf{u}))^\alpha \leq \widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u})(-\ln \widehat{\mathbf{C}}_{\mathbf{Z}}(\mathbf{u}))^\alpha. \tag{2.29}$$

Now, by integrating both sides of the inequality in (2.29), the required result follows, which completes the proof of the proposition. □

Proposition 2.4. *Suppose three random vectors* $\mathbf{X}$, $\mathbf{Y}$ *and* $\mathbf{Z}$ *have survival copulas* $\widehat{C}_{\mathbf{X}}$, $\widehat{C}_{\mathbf{Y}}$ *and* $\widehat{C}_{\mathbf{Z}}$, *respectively. Let further* $\widehat{C}_{\mathbf{X}} \prec \widehat{C}_{\mathbf{Y}} \prec \widehat{C}_{\mathbf{Z}}$. *Then,*

$$K_\alpha(\mathbf{X}||\mathbf{Y}) + \mathbf{K}_\alpha(\mathbf{Y}||\mathbf{Z}) \geq \mathbf{K}_\alpha(\mathbf{X}||\mathbf{Z}).$$

Proof: We have

$$
\begin{aligned}
&K_\alpha(\mathbf{X}||\mathbf{Y})+\mathbf{K}_\alpha(\mathbf{Y}||\mathbf{Z})\\
&=\frac{1}{\Gamma(1+\alpha)}\int_0^1\cdots\int_0^1\left\{\widehat{C}_{\mathbf{X}}(\mathbf{u})(-\ln\widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u}))^\alpha+\widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u})(-\ln\widehat{\mathbf{C}}_{\mathbf{Z}}(\mathbf{u}))^\alpha\right\}d\mathbf{u}\\
&\geq\frac{1}{\Gamma(1+\alpha)}\int_0^1\cdots\int_0^1\left\{\widehat{C}_{\mathbf{X}}(\mathbf{u})(-\ln\widehat{\mathbf{C}}_{\mathbf{Z}}(\mathbf{u}))^\alpha+\widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u})(-\ln\widehat{\mathbf{C}}_{\mathbf{Z}}(\mathbf{u}))^\alpha\right\}d\mathbf{u}\\
&\geq\frac{1}{\Gamma(1+\alpha)}\int_0^1\cdots\int_0^1\left\{\widehat{C}_{\mathbf{X}}(\mathbf{u})(-\ln\widehat{\mathbf{C}}_{\mathbf{Z}}(\mathbf{u}))^\alpha+\widehat{\mathbf{C}}_{\mathbf{X}}(\mathbf{u})(-\ln\widehat{\mathbf{C}}_{\mathbf{Z}}(\mathbf{u}))^\alpha\right\}d\mathbf{u}\\
&\geq\frac{1}{\Gamma(1+\alpha)}\int_0^1\cdots\int_0^1\widehat{C}_{\mathbf{X}}(\mathbf{u})(-\ln\widehat{\mathbf{C}}_{\mathbf{Z}}(\mathbf{u}))^\alpha\mathbf{du}=\mathbf{K}_\alpha(\mathbf{X}||\mathbf{Z}),
\end{aligned}
\tag{2.30}
$$

where the first inequality follows from $\widehat{C}_{\mathbf{Y}}\prec\widehat{C}_{\mathbf{Z}}$ and the second inequality follows from $\widehat{C}_{\mathbf{X}}\prec\widehat{C}_{\mathbf{Y}}$. This completes the proof of the proposition. □

Proposition 2.5. *Suppose three random vectors* $\mathbf{X}$, $\mathbf{Y}$ *and* $\mathbf{Z}$ *have survival copulas* $\widehat{C}_{\mathbf{X}}$, $\widehat{C}_{\mathbf{Y}}$ *and* $\widehat{C}_{\mathbf{Z}}$*, respectively. Let further* $\widehat{C}_{\mathbf{Z}}\prec\widehat{C}_{\mathbf{Y}}\prec\widehat{C}_{\mathbf{X}}$*. Then,*

$$K_\alpha(\mathbf{X}||\mathbf{Y})+\mathbf{K}_\alpha(\mathbf{Y}||\mathbf{Z})\leq\mathbf{K}_\alpha(\mathbf{X}||\mathbf{Z}).$$

Proof: The proof follows along the same lines as those used in Proposition 2.4, and is therefore omitted for brevity. □

Below, we present a bound for the inaccuracy measure between a survival copula function $\widehat{C}$ and the weighted arithmetic mean of m survival copulas $\widehat{C}_1,\ldots,\widehat{C}_m$. Let $\widehat{C}_1,\ldots,\widehat{C}_m$ be m bivariate copulas. Then, their weighted arithmetic mean is given by

$$\widehat{C}^*(\mathbf{u})=\sum_{\mathbf{k=1}}^{\mathbf{m}}\mathbf{l_k}\widehat{\mathbf{C}}_{\mathbf{k}}(\mathbf{u}),\tag{2.31}$$

where $l_k\in J$ and $\sum_{k=1}^m l_k=1$. Note that the weighted arithmetic mean of m copulas provides a valid copula (see Cuadras, 2009). In order to prove the next proposition, we require the following lemma.

Lemma 2.1. *The function* $h(u)=(-\ln u)^a$ *is convex with respect to* $u\in(0,1)$*, for* $a\geq1$.

Proof: The second derivative of $h(u)$ with respect to u is given by

$$\frac{d^2h(u)}{du^2}=\frac{(-\ln u)^{a-1}}{u^2}[a(a-1)(-\ln u)^{-1}+1]\geq0,\ \text{for } a\geq1.\tag{2.32}$$

Hence, the lemma. □

Proposition 2.6. *Consider m survival copulas $\widehat{C}_1, \ldots, \widehat{C}_m$. Let*

$$\widehat{C}^*(\mathbf{u}) = \sum_{\mathbf{k=1}}^{\mathbf{m}} \mathbf{l_k} \widehat{\mathbf{C}}_{\mathbf{k}}(\mathbf{u}),$$

where $l_k \in J$, $k = 1, \ldots, m$, and $\sum_{k=1}^{m} l_k = 1$. Further, let $\widehat{C}$ be any copula. Then, for $\alpha \geq 1$,

$$K_\alpha(\widehat{C}||\widehat{C}^*) \leq \sum_{k=1}^{m} l_k K_\alpha(\widehat{C}||\widehat{C}_k). \tag{2.33}$$

Proof: Under the assumption made, from Lemma 2.1, we have

$$\left(-\ln\left(\sum_{k=1}^{n} l_k \widehat{C}_k(\mathbf{u})\right)\right)^\alpha \leq \sum_{k=1}^{m} l_k \left(-\ln \widehat{C}_k(\mathbf{u})\right)^\alpha. \tag{2.34}$$

Utilizing this inequality, we obtain from Definition 2.2 that

$$\begin{aligned} K_\alpha(\widehat{C}||\widehat{C}^*) &= \frac{1}{\Gamma(1+\alpha)} \int_0^1 \cdots \int_0^1 \widehat{C}(\mathbf{u}) \left(-\ln \widehat{\mathbf{C}}^*(\mathbf{u})\right)^\alpha \mathbf{du} \\ &\leq \frac{1}{\Gamma(1+\alpha)} \int_0^1 \cdots \int_0^1 \widehat{C}(\mathbf{u}) \sum_{\mathbf{k=1}}^{\mathbf{m}} \mathbf{l_k} \left(-\ln \widehat{\mathbf{C}}_{\mathbf{k}}(\mathbf{u})\right)^\alpha \mathbf{du} \\ &= \sum_{k=1}^{m} \frac{l_k}{\Gamma(1+\alpha)} \int_0^1 \cdots \int_0^1 \widehat{C}(\mathbf{u}) \left(-\ln \widehat{\mathbf{C}}_{\mathbf{k}}(\mathbf{u})\right)^\alpha \mathbf{du} \\ &= \sum_{k=1}^{m} l_k K_\alpha(\widehat{C}||\widehat{C}_k). \end{aligned} \tag{2.35}$$

This completes the proof of the proposition. □

For m survival copulas $\widehat{C}_1, \ldots, \widehat{C}_m$, the weighted geometric mean is given by

$$\widehat{C}_G(\mathbf{u}) = \prod_{\mathbf{k=1}}^{\mathbf{m}} \left(\widehat{\mathbf{C}}_{\mathbf{k}}(\mathbf{u})\right)^{\mathbf{l_k}}, \tag{2.36}$$

where $l_k \in J$, $k = 1, \ldots, m$, and $\sum_{k=1}^{m} l_k = 1$. The weighted geometric mean of m copulas may not be a valid copula (Cuadras, 2009). However, in this case, we get a valid copula with an additional condition (Cuadras, 2009; Zhang et al., 2013). The proposition below provides an upper bound for the fractional-order inaccuracy measure associated with the weighted geometric mean of survival copulas.

Proposition 2.7. *Let $\widehat{C}_1, \ldots, \widehat{C}_m$ be m survival copulas. Further, let $\widehat{C}_G(\mathbf{u}) = \prod_{\mathbf{k=1}}^{\mathbf{m}} \left(\widehat{\mathbf{C}}_{\mathbf{k}}(\mathbf{u})\right)^{\mathbf{l_k}}$ be their weighted geometric mean. Then, for any survival copula $\widehat{C}$, we have*

$$K_\alpha(\widehat{C}_G||\widehat{C}) \leq \left(\Gamma(1+\alpha)\right)^{l_k - 1} \prod_{k=1}^{m} \left(K_\alpha(\widehat{C}_k||\widehat{C})\right)^{l_k}. \tag{2.37}$$

Proof: For a function $h : J^m \to R^+$ and $s \neq 0$, we get

$$\int_0^1 \dots \int_0^1 h(\mathbf{u}) \left(\widehat{\mathbf{C}}_{\mathbf{G}}(\mathbf{u})\right)^{\mathbf{s}} \mathbf{du} = \int_{\mathbf{0}}^{\mathbf{1}} \dots \int_{\mathbf{0}}^{\mathbf{1}} \prod_{\mathbf{k=1}}^{\mathbf{m}} (\mathbf{h}(\mathbf{u}))^{\mathbf{l_k}} \left(\widehat{\mathbf{C}}_{\mathbf{k}}(\mathbf{u})\right)^{\mathbf{sl_k}} \mathbf{du}. \tag{2.38}$$

Now, using generalized Holder's inequality, we obtain from (2.38) that

$$\int_0^1 \dots \int_0^1 h(\mathbf{u}) \left(\widehat{\mathbf{C}}_{\mathbf{G}}(\mathbf{u})\right)^{\mathbf{s}} \mathbf{du} \leq \prod_{\mathbf{k=1}}^{\mathbf{m}} \left(\int_{\mathbf{0}}^{\mathbf{1}} \dots \int_{\mathbf{0}}^{\mathbf{1}} \mathbf{h}(\mathbf{u}) \left(\widehat{\mathbf{C}}_{\mathbf{k}}(\mathbf{u})\right)^{\mathbf{s}} \mathbf{du}\right)^{\mathbf{l_k}}. \tag{2.39}$$

Further, from Definition 2.2, we have

$$K_\alpha(\widehat{C}_G||\widehat{C}) = \frac{1}{\Gamma(1+\alpha)} \int_0^1 \dots \int_0^1 \widehat{C}_G(\mathbf{u}) \left(-\ln \widehat{\mathbf{C}}(\mathbf{u})\right)^{\alpha} \mathbf{du}, \ \alpha > \mathbf{0}. \tag{2.40}$$

Substituting $h(\mathbf{u}) = \left(-\ln \widehat{\mathbf{C}}(\mathbf{u})\right)^{\alpha}$ and $s = 1$ in (2.39), we finally obtain

$$\begin{aligned} K_\alpha(\widehat{C}_G||\widehat{C}) &\leq \left(\Gamma(1+\alpha)\right)^{l_k - 1} \prod_{k=1}^{m} \left(\frac{1}{\Gamma(1+\alpha)} \int_0^1 \dots \int_0^1 \left(-\ln \widehat{C}(\mathbf{u})\right)^{\alpha} \widehat{C}_k(\mathbf{u}) \mathbf{du}\right)^{l_k} \\ &= \left(\Gamma(1+\alpha)\right)^{l_k - 1} \prod_{k=1}^{m} \left(K_\alpha(\widehat{C}_k||\widehat{C})\right)^{l_k}. \end{aligned} \tag{2.41}$$

This completes the proof of the proposition. □

Suppose $\mathbf{X} = (X_1, \dots, X_n)$ have respective CDFs $F_i(\cdot)$ and joint distribution function $H(\cdot, \dots, \cdot)$ with corresponding copula $C_{\mathbf{X}}$ and survival copula $\widehat{C}_{\mathbf{X}}$. Then, X_i are positively (negatively) quadrant dependent PQD (NQD) if $H(x_1, \dots, x_n) \geq (\leq) \prod_{i=1}^n F_i(x_i)$ for all $(x_1, \dots, x_n) \in R^n$ or equivalently,

$$C_X(\mathbf{u}) \text{ or } \widehat{C}_{\mathbf{X}}(\mathbf{u}) \geq (\leq) \prod_{i=1}^{n} u_i, \quad \text{for all } u_i \in [0,1], \ i = 1, \dots, n.$$

For more details one may refer to Nelsen (2006). Now, we discuss results of the FGSCI relating to the PQD or NQD.

Proposition 2.8. *Suppose two random vectors $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ are with corresponding survival copulas $\widehat{C}_{\mathbf{X}}$ and $\widehat{C}_{\mathbf{Y}}$, respectively.*

(A) *For PQD random vector $\boldsymbol{X}$ and NQD random vector $\boldsymbol{Y}$, then*

$$K_\alpha(\mathbf{X}||\mathbf{Y}) \geq \frac{n}{2^{n+1}\Gamma(1+\alpha)};$$

(B) *For NQD random vector $\boldsymbol{X}$ and PQD random vector $\boldsymbol{Y}$, then*

$$K_\alpha(\mathbf{X}||\mathbf{Y}) \leq \frac{n}{2^{n+1}\Gamma(1+\alpha)}.$$

Proof: (A) Since $\mathbf{X}$ and $\mathbf{Y}$ have PQD and NQD, respectively, we can write for all $u_i \in [0,1],\ i = 1, \ldots, n,$

$$\widehat{C}_{\mathbf{X}}(\mathbf{u}) \geq \prod_{i=1}^{n} u_i \quad \text{and} \quad \widehat{C}_{\mathbf{Y}}(\mathbf{u}) \leq \prod_{i=1}^{n} u_i. \tag{2.42}$$

Now,

$$\begin{aligned}
&\widehat{C}_{\mathbf{Y}}(\mathbf{u}) \leq \prod_{i=1}^{n} u_i \\
\Longrightarrow\ &\Big\{-\ln \widehat{C}_{\mathbf{Y}}(\mathbf{u})\Big\}^{\alpha} \geq \left\{\sum_{i=1}^{n}(-\ln u_i)\right\}^{\alpha} \geq \sum_{i=1}^{n}(-\ln u_i)^{\alpha} \geq \sum_{i=1}^{n}(-\ln u_i).
\end{aligned} \tag{2.43}$$

Using (2.42) and (2.43) in (2.5), we have

$$\begin{aligned}
K_\alpha(\mathbf{X}||\mathbf{Y}) &\geq \frac{1}{\Gamma(1+\alpha)} \int_0^1 \cdots \int_0^1 (u_1 \ldots u_n)\{(-\ln u_1) + \cdots + (-\ln u_n)\} du_1 \cdots du_n \\
&= \frac{1}{\Gamma(1+\alpha)} \frac{n}{2^{n+1}},
\end{aligned} \tag{2.44}$$

completing the proof of Part (A).

(B) The proof is quite similar to the proof of Part (A), and so do not present it here for brevity. Hence, the theorem. □

3. JEFFREYS' INACCURACY BASED ON FGSCI/FGCI MEASURES

In this section, we propose Jeffreys' inaccuracy between two survival copula functions by combining two directional FGSCI measures. The idea is based on Jeffreys' distance or Jeffreys' divergence (Jeffreys, 1946). The proposed inaccuracy is defined as the average of the two FGSCI measures, making it a symmetric measure. Let $\mathbf{X} = (\mathbf{X_1}, \ldots, \mathbf{X_n})$ and $\mathbf{Y} = (\mathbf{Y_1}, \ldots, \mathbf{Y_n})$ be two random vectors with survival copulas $\widehat{C}_{\mathbf{X}}$ and $\widehat{C}_{\mathbf{Y}}$, and distribution copulas $C_{\mathbf{X}}$ and $C_{\mathbf{Y}}$, respectively.

Definition 3.1. The Jeffreys' inaccuracy between two survival copula functions $\widehat{C}_{\mathbf{X}}$ and $\widehat{C}_{\mathbf{Y}}$ is given by

$$\begin{aligned}
D_\alpha(\widehat{C}_{\mathbf{X}}||\widehat{C}_{\mathbf{Y}}) &= \frac{1}{2}\left[K_\alpha(\mathbf{X}||\mathbf{Y}) + \mathbf{K}_\alpha(\mathbf{X}||\mathbf{Y})\right] \\
&= \frac{1}{2\Gamma(\alpha+1)} \int_0^1 \cdots \int_0^1 \left[\widehat{C}_{\mathbf{X}}(\mathbf{u})(-\ln \widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u}))^{\alpha} + \widehat{\mathbf{C}}_{\mathbf{Y}}(\mathbf{u})(-\ln \widehat{\mathbf{C}}_{\mathbf{X}}(\mathbf{u}))^{\alpha}\right] d\mathbf{u}
\end{aligned} \tag{3.1}$$

The following observations follow readily from the above definition:

(a) $D_\alpha(\widehat{C}_{\mathbf{X}}||\widehat{C}_{\mathbf{Y}}) = D_\alpha(\widehat{C}_{\mathbf{Y}}||\widehat{C}_{\mathbf{X}})$, that is, the Jeffreys' inaccuracy is symmetric;

(b) When $\widehat{C}_{\mathbf{X}} = \widehat{C}_{\mathbf{Y}}$, (3.1) reduces to the fractional order generalized survival copula entropy;

(c) When $\widehat{C}_{\mathbf{X}} = \widehat{C}_{\mathbf{Y}}$ and $\alpha = 1$, (3.1) reduces to the survival copula entropy.

Next, we obtain bounds for Jeffreys' inaccuracy defined in (3.1).

Proposition 3.1. *For two random vectors* $\mathbf{X}$ *and* $\mathbf{Y}$, *we have*

$$D_\alpha(\widehat{C}_{\mathbf{X}}||\widehat{C}_{\mathbf{Y}}) \begin{cases} \leq \frac{1}{2\Gamma(1+\alpha)}\left[(K(\mathbf{X}||\mathbf{Y}))^\alpha + (\mathbf{K}(\mathbf{Y}||\mathbf{X}))^\alpha\right] \ \textit{if } \alpha < 1 \\ \geq \frac{1}{2\Gamma(1+\alpha)}\left[(K(\mathbf{X}||\mathbf{Y}))^\alpha + (\mathbf{K}(\mathbf{Y}||\mathbf{X}))^\alpha\right] \ \textit{if } \alpha > 1, \end{cases}$$

where $K(\mathbf{X}||\mathbf{Y})$ *is as defined in Proposition 2.1.*

Proof: The proof is similar to that of Proposition 2.1, and is therefore omitted for brevity. □

Proposition 3.2. *Consider* m *survival copulas* $\widehat{C}_1, \ldots, \widehat{C}_m$. *Let*

$$\widehat{C}^*(\mathbf{u}) = \sum_{\mathbf{k=1}}^{\mathbf{m}} \mathbf{l_k}\widehat{\mathbf{C}}_{\mathbf{k}}(\mathbf{u}),$$

where $l_k \in J$, $k = 1, \ldots, m$, *and* $\sum_{k=1}^m l_k = 1$. *Further, let* $\widehat{C}$ *be any copula. Then, for* $\alpha \geq 1$,

$$D_\alpha(\widehat{C}||\widehat{C}^*) \leq \frac{1}{2}\sum_{k=1}^m l_k \left\{K_\alpha(\widehat{C}||\widehat{C}_k) + K_\alpha(\widehat{C}_k||\widehat{C})\right\}. \tag{3.2}$$

Proof: We have

$$D_\alpha(\widehat{C}||\widehat{C}^*) = \frac{1}{2}\left[K_\alpha(\widehat{C}||\widehat{C}^*) + K_\alpha^*(\widehat{C}^*||\widehat{C})\right]. \tag{3.3}$$

Further,

$$\begin{aligned} K_\alpha^*(\widehat{C}^*||\widehat{C}) &= \frac{1}{\Gamma(1+\alpha)}\int_0^1 \cdots \int_0^1 \widehat{C}^*(\mathbf{u})\left(-\ln \widehat{\mathbf{C}}(\mathbf{u})\right)^\alpha \mathbf{du} \\ &= \frac{1}{\Gamma(1+\alpha)}\int_0^1 \cdots \int_0^1 \left(\sum_{k=1}^m l_k\widehat{C}_k(\mathbf{u})\right)\left(-\ln \widehat{C}(\mathbf{u})\right)^\alpha d\mathbf{u} \\ &= \sum_{k=1}^m \frac{l_k}{\Gamma(1+\alpha)}\int_0^1 \cdots \int_0^1 \widehat{C}_k(\mathbf{u})\left(-\ln \widehat{\mathbf{C}}(\mathbf{u})\right)^\alpha \mathbf{du} \\ &= \sum_{k=1}^m l_k K_\alpha(\widehat{C}_k||\widehat{C}). \end{aligned} \tag{3.4}$$

Thus, the required result in (3.2) follows readily from (3.3), (3.4) and Proposition 2.6. This completes the proof of the proposition. □

The Jeffreys' inaccuracy between two copula functions can be proposed in a manner analogous to Definition 3.1.

Definition 3.2. The Jeffreys' inaccuracy between two copula functions $C_{\mathbf{X}}$ and $C_{\mathbf{Y}}$ is given by

$$\begin{aligned} D_\alpha(C_{\mathbf{X}}||C_{\mathbf{Y}}) &= \frac{1}{2}\left[K^*_\alpha(\mathbf{X}||\mathbf{Y}) + \mathbf{K}^*_\alpha(\mathbf{Y}||\mathbf{X})\right] \\ &= \frac{1}{2\Gamma(\alpha+1)}\int_0^1 \cdots \int_0^1 \left[C_{\mathbf{X}}(\mathbf{u})(-\ln \mathbf{C}_{\mathbf{Y}}(\mathbf{u}))^\alpha + \mathbf{C}_{\mathbf{Y}}(\mathbf{u})(-\ln \mathbf{C}_{\mathbf{X}}(\mathbf{u}))^\alpha\right] d\mathbf{u}, \end{aligned} \tag{3.5}$$

where $\alpha > 0$.

Remark 3.1. Results similar to those for Jeffreys' inaccuracy between two survival copula functions can be obtained for the measure defined in (3.5) as well. But, we refrain from presenting them here, for the sake of brevity.

4. SEMIPARAMETRIC ESTIMATION OF FGSCI AND FGCI

In this section, we introduce semiparametric estimators for the FGSCI and FGCI measures. The semiparametric estimation technique is particularly important when marginal distributions are complex or unknown, and accurate modelling of dependence is critical for decision-making, and in addition robustness to misspecification of marginal distributions is essential. We first discuss the method of semiparametric copula estimation. In this regard, we have presented as Algorithm for the semiparametric estimation below for bivariate copula functions.

Algorithm 1: Semiparametric estimation of copula functions

Step-1: Generate n observations $(U_{1,j}, U_{2,j})$ and $(V_{1,j}, V_{2,j})$, $j = 1, \ldots, n$, from the bivariate Frank copula with parameter θ and Gumbel copula with parameter ϕ, respectively;
Step-2: Convert n observations into pseudo-observations for $i = 1, 2,\ j = 1, \ldots, n$:

$$\widehat{u}_{ij} = \frac{\text{rank}(U_{ij})}{n+1}, \qquad \widehat{v}_{ij} = \frac{\text{rank}(V_{ij})}{n+1};$$

Step-3: Next, construct pseudo-data matrix $\widehat{U}_j = (\widehat{u}_{1j}, \widehat{u}_{2j})$ and $\widehat{V}_j = (\widehat{v}_{1j}, \widehat{v}_{2j})$;
Step-4: Estimate the parameters θ and ϕ of the Frank and Gumbel copulas using Maximum Pseudo Likelihood (MPL) method as

$$\widehat{\theta} = arg\max \sum_{j=1}^{n} \log c_F(\widehat{U}_j; \theta), \qquad \widehat{\phi} = arg\max \sum_{j=1}^{n} \log c_G(\widehat{V}_j; \theta),$$

where c_F and c_G are the density copula functions of Frank and Gumbel copula functions.

Algorithm 2: Semiparametric estimation of survival copula functions

Step-1: Generate n observations $(\overline{U}_{1,j}, \overline{U}_{2,j})$ and $(\overline{U}_{1,j}, \overline{U}_{2,j})$, $j = 1, \ldots, n$, from the bivariate survival Frank copula with parameter θ and survival Gumbel copula with parameter ϕ, respectively;
Step-2: Convert n observations into pseudo-observations, for $i = 1, 2$, $j = 1, \ldots, n$:

$$\widehat{\overline{u}}_{ij} = \frac{rank(\overline{U}_{ij})}{n+1}, \quad \widehat{\overline{v}}_{ij} = \frac{rank(\overline{V}_{ij})}{n+1};$$

Step-3: Construct pseudo-data matrix $\widehat{\overline{U}}_j = (\widehat{\overline{u}}_{1j}, \widehat{\overline{u}}_{2j})$ and $\widehat{\overline{V}}_j = (\widehat{\overline{v}}_{1j}, \widehat{\overline{v}}_{2j})$;
Step-4: Estimate the parameters θ and ϕ of the Frank and Gumbel copulas using Maximum Pseudo Likelihood (MPL) method as

$$\widehat{\theta} = arg \max \sum_{j=1}^{n} \log c_F(\widehat{\overline{U}}_j; \theta), \quad \widehat{\phi} = arg \max \sum_{j=1}^{n} \log c_G(\widehat{\overline{V}}_j; \theta),$$

where c_F and c_G are the density copula functions of Frank and Gumbel copula functions.

Definition 4.1. Suppose $\widehat{C}_{\mathbf{X}}(\cdot,\cdot)$, $\widehat{C}_{\mathbf{Y}}(\cdot,\cdot)$ and $C_{\mathbf{X}}(\cdot,\cdot)$, $C_{\mathbf{Y}}(\cdot,\cdot)$ are the bivariate survival copula and distribution copula functions of $\mathbf{X}$ and $\mathbf{Y}$, respectively. Then, the semiparametric estimators of FGSCI and FGCI measures, for $\alpha > 0$, are

$$\widehat{K_\alpha}(\mathbf{X}||\mathbf{Y}) = \frac{1}{\Gamma(1+\alpha)} \int_0^1 \int_0^1 \widehat{C}^{\delta}_{\mathbf{X}}(u,v) \big\{ -\ln \widehat{C}^{\delta}_{\mathbf{Y}}(u,v) \big\}^{\alpha} dudv \tag{4.1}$$

and

$$\widehat{K^*_\alpha}(\mathbf{X}||\mathbf{Y}) = \frac{1}{\Gamma(1+\alpha)} \int_0^1 \int_0^1 C^{\delta}_{\mathbf{X}}(u,v) \big\{ -\ln C^{\delta}_{\mathbf{Y}}(u,v) \big\}^{\alpha} dudv, \tag{4.2}$$

respectively, where $\widehat{C}^{\delta}_{\mathbf{X}}(\cdot,\cdot)$, $\widehat{C}^{\delta}_{\mathbf{Y}}(\cdot,\cdot)$, $C^{\delta}_{\mathbf{X}}(\cdot,\cdot)$ and $C^{\delta}_{\mathbf{Y}}(\cdot,\cdot)$ are the semiparametric estimators of $\widehat{C}_{\mathbf{X}}(\cdot,\cdot)$, $\widehat{C}_{\mathbf{Y}}(\cdot,\cdot)$, $C_{\mathbf{X}}(\cdot,\cdot)$ and $C_{\mathbf{Y}}(\cdot,\cdot)$, respectively.

Next, we carry out a Monte Carlo simulation study for examining the performance of the semiparametric estimators in (4.1) and (4.2), respectively. For this purpose, we use bivariate Frank and Gumbel copulas, and Frank and Gumbel survival copulas for FGCI and FGSCI measures, respectively. The survival Frank and Gumbel copulas are, respectively, given by

$$\widehat{C}_{\mathbf{X}}(u,v) = u + v - 1 - \frac{1}{\theta} \log \left[1 + \frac{(e^{-\theta(1-u)})(e^{-\theta(1-v)})}{e^{-\theta} - 1} \right], \ \theta \geq 1, \tag{4.3}$$

and

$$\widehat{C}_{\mathbf{Y}}(u,v) = u + v - 1 + \exp\Big\{ - \big((-\log(1-u))^{\phi} + (-\log(1-v))^{\phi} \big)^{\frac{1}{\phi}} \Big\}, \ \phi \geq 1. \tag{4.4}$$

As mentioned earlier, the marginal distribution and survival functions are estimated by the corresponding empirical versions. Further, the method of maximum psuedo-likelihood (MPL) is used for estimating the parameters of copulas and the survival copulas. The estimated parameters so obtained are denoted by $\widehat{\phi}$ and $\widehat{\theta}$. The semiparametric copula estimators are obtained as $C^{\delta}_{\mathbf{X}}(\cdot,\cdot)$ and $C^{\delta}_{\mathbf{Y}}(\cdot,\cdot)$, and the semiparametric survival copula estimators are obtained as $\widehat{C}^{\delta}_{\mathbf{X}}(\cdot,\cdot)$ and $\widehat{C}^{\delta}_{\mathbf{Y}}(\cdot,\cdot)$. 500 replications with sample sizes $n = 100, 400$ and 600 are used in the Monte Carlo simulation study for obtaining the values of standard deviation (SD), absolute bias (AB) and mean squared error (MSE). The SD, AB and MSE of the semiparametric FGSCI and FGCI estimators are computed for different choices of n, ϕ, α and θ, and these are all reported in Tables 1 and 2, respectively. For the involved simulations, we used the "R-software". The numerical values in Tables 1 and 2 suggest that the proposed estimators in (4.1) and (4.2) are consistent since the values of AB, MSE and SD all decrease when n increases.

Table 1: The SD, AB and MSE of the proposed semiparametric estimator of FSCI measure in (4.1) for different choices of θ, ϕ, α and n.

	$\phi=2,\ \alpha=1.5$				$\theta=3,\ \alpha=1.5$				$\theta=2,\ \phi=3$		
θ	**n**	**SD**	**AB (MSE)**	ϕ	n	**SD**	**AB (MSE)**	α	n	**SD**	**AB (MSE)**
	100	0.010421	0.004398 (0.000128)		100	0.012043	0.005616 (0.000177)		100	0.009146	0.002783 (9.1×10^{-5})
1.0	400	0.00538	0.003985 (4.5×10^{-5})	1.5	400	0.006088	0.005118 (6.3×10^{-5})	0.5	400	0.004696	0.001967 (2.6×10^{-5})
	600	0.004554	0.003913 (3.6×10^{-5})		600	0.005164	0.005122 (5.3×10^{-5})		600	0.003938	0.001890 (1.9×10^{-5})
	100	0.010237	0.004670 (0.000127)		100	0.007669	0.003879 (7.4×10^{-5})		100	0.008502	0.003694 (8.6×10^{-5})
1.5	400	0.005288	0.004193 (4.6×10^{-5})	3.0	400	0.003924	0.003207 (2.6×10^{-5})	1.5	400	0.004367	0.002993 (2.8×10^{-5})
	600	0.004473	0.004132 (3.7×10^{-5})		600	0.003306	0.003197 (2.1×10^{-5})		600	0.003677	0.002946 (2.2×10^{-5})
	100	0.00966	0.005003 (0.000118)		100	0.00717	0.002998 (6.0×10^{-5})		100	0.005803	0.002575 (4.0×10^{-5})
2.5	400	0.004966	0.004554 (4.5×10^{-5})	4.0	400	0.003646	0.002170 (1.8×10^{-5})	2.5	400	0.002979	0.002072 (1.3×10^{-5})
	600	0.004204	0.004544 (3.8×10^{-5})		600	0.003066	0.002127 (1.4×10^{-5})		600	0.002509	0.002038 (1.0×10^{-5})
	100	0.008768	0.005288 (0.000105)		100	0.006925	0.002380 (5.4×10^{-5})		100	0.003397	0.001394 (1.3×10^{-5})
4.0	400	0.004447	0.004990 (4.5×10^{-5})	5.0	400	0.003532	0.001550 (1.5×10^{-5})	3.5	400	0.001743	0.001079 (4.0×10^{-6})
	600	0.003726	0.005021 (3.9×10^{-5})		600	0.002967	0.001482 (1.1×10^{-5})		600	0.001468	0.001055 (3.0×10^{-6})
	100	0.008237	0.005305 (9.6×10^{-5})		100	0.006832	0.002015 (5.1×10^{-5})		100	0.001328	0.000442 (2.0×10^{-5})
5.0	400	0.004141	0.005200 (4.4×10^{-5})	6.0	400	0.003481	0.001150 (1.3×10^{-5})	5.0	400	0.000681	0.000309 (1.0×10^{-6})
	600	0.003442	0.005271 (4.0×10^{-5})		600	0.002918	0.001078 (1.0×10^{-5})		600	0.000573	0.000297 (4.0×10^{-7})

Table 2: The SD, AB and MSE of the proposed semiparametric estimator of FGCI measure in (4.2) for different choices of θ, ϕ, α and n.

	$\phi = 2,\ \alpha = 1.5$				$\theta = 3,\ \alpha = 1.5$				$\theta = 2,\ \phi = 3$		
θ	**n**	**SD**	**AB (MSE)**	ϕ	n	**SD**	**AB (MSE)**	α	n	**SD**	**AB (MSE)**
	100	0.010879	0.004621 (0.000140)		100	0.012596	0.00582 (0.000193)		100	0.009159	0.002592 (9.1×10^{-5})
1.0	400	0.005616	0.004189 (4.9×10^{-5})	1.5	400	0.006391	0.005345 (6.9×10^{-5})	0.5	400	0.004702	0.001744 (2.5×10^{-5})
	600	0.004755	0.004113 (4.0×10^{-5})		600	0.005422	0.005353 (5.8×10^{-5})		600	0.00394	0.001660 (1.8×10^{-5})
	100	0.010715	0.004930 (0.000139)		100	0.007967	0.004161 (8.1×10^{-5})		100	0.008778	0.003905 (9.2×10^{-5})
1.5	400	0.005379	0.004293 (4.7×10^{-5})	3.0	400	0.004074	0.003471 (2.9×10^{-5})	1.5	400	0.004507	0.003182 (3.0×10^{-5})
	600	0.004683	0.004373 (4.1×10^{-5})		600	0.003434	0.003464 (2.4×10^{-5})		600	0.003796	0.003136 (2.4×10^{-5})
	100	0.010172	0.005326 (0.000132)		100	0.007352	0.003201 (6.4×10^{-5})		100	0.006349	0.003262 (5.1×10^{-5})
2.5	400	0.00523	0.004870 (5.1×10^{-5})	4.0	400	0.003734	0.002349 (1.9×10^{-5})	2.5	400	0.003254	0.002752 (1.8×10^{-5})
	600	0.004428	0.004861 (4.3×10^{-5})		600	0.003142	0.002306 (1.5×10^{-5})		600	0.002743	0.002728 (1.5×10^{-5})
	100	0.009326	0.005687 (0.000119)		100	0.007042	0.002524 (5.6×10^{-5})		100	0.003956	0.002162 (2.0×10^{-5})
4.0	400	0.00473	0.005398 (5.2×10^{-5})	5.0	400	0.003588	0.001674 (1.6×10^{-5})	3.5	400	0.002022	0.001837 (7.0×10^{-6})
	600	0.003964	0.005435 (4.5×10^{-5})		600	0.003016	0.001606 (1.2×10^{-5})		600	0.001705	0.001823 (6.0×10^{-6})
	100	0.008821	0.005736 (0.000111)		100	0.006916	0.002125 (5.2×10^{-5})		100	0.001664	0.000936 (4.0×10^{-6})
5.0	400	0.004436	0.005656 (5.2×10^{-5})	6.0	400	0.00352	0.001241 (1.4×10^{-5})	5.0	400	0.000848	0.000790 (1.0×10^{-6})
	600	0.003687	0.005736 (4.6×10^{-5})		600	0.002953	0.001168 (1.0×10^{-5})		600	0.000715	0.000783 (1.0×10^{-6})

5. AN APPLICATION TO MODEL SELECTION

Here, we discuss an application of the FGCI measure in selecting best model (copula). The "Pima Indians Diabetes" data set with 724 entries is used for this purpose. These data were collected by NIDDK. During the data collection, mainly the ladies of 21 years old and above, who were of Pima Indian descent and living around Phoenix, Arizona, were considered. These data are available from the *R* software within the **pdp** package. This real data set has been analyzed recently by Arshad et al. (2024). They obtained p-values and the estimated values of the parameters for Clayton, Frank, Gumbel-Hougaarad, Joe, Normal and product copulas. In their study, they considered a trivariate data set with 724 entries considering the variables "glucose", "pressure" and "mass", which represent plasma glucose concentration, diastolic blood pressure (mm Hg) and body mass index, respectively. In a similar vein, we consider here Frank, Gumbel-Hougaard, Joe and product copulas for the purpose of model selection. The p-values and estimated values of the parameters of these copulas are presented in Table 3 (also see Arshad et al., 2024).

Table 3: Different copulas fitted with the real data set.

Copula	Parameter	p-value
Frank	1.3776	0.488
Gumbel-Hougaarad	1.1542	0.036
Joe	1.1977	0
Product	0	0

From Table 3, it is clear that the Frank copula fits better than all other copulas. Next, we obtain the FGCI measure between Frank ($\mathbf{X}$) and Gumbel-Hougaard ($\mathbf{Y}$) copulas (denoted by $K_\alpha^*(\mathbf{X}||\mathbf{Y})$), Frank ($\mathbf{X}$) and Joe ($\mathbf{Z}$) copulas (denoted by $K_\alpha^*(\mathbf{X}||\mathbf{Z})$), and Frank ($\mathbf{X}$) and Product ($\mathbf{W}$) copulas (denoted by $K_\alpha^*(\mathbf{X}||\mathbf{W})$). For illustrative purposes, we have chosen $\alpha = 0.5,\ 3.0$ and 4.0. The values of the FGCI measures so computed are reported in Table 4. From Table 4, we observe that the values of the FGCI measures, for $\alpha > 0$, are

$$K_\alpha^*(\mathbf{X}||\mathbf{Y}) < K_\alpha^*(\mathbf{X}||\mathbf{Z}) < K_\alpha^*(\mathbf{X}||\mathbf{W}),$$

as one would expect. These results do reveal that FGCI is a measure that could effectively be used as a model (copula) selection criteria.

Table 4: Values of the FGCI measure for different combinations of the copulas.

α	FGCI Measure	Value	α	FGCI Measure	Value	α	FGCI Measure	Value
	$K_\alpha^*(\mathbf{X}\|\|\mathbf{Y})$	0.2156361		$K_\alpha^*(\mathbf{X}\|\|\mathbf{Y})$	0.1672478		$K_\alpha^*(\mathbf{X}\|\|\mathbf{Y})$	0.1193678
0.9	$K_\alpha^*(\mathbf{X}\|\|\mathbf{Z})$	0.221795	3.0	$K_\alpha^*(\mathbf{X}\|\|\mathbf{Z})$	0.1910817	4.0	$K_\alpha^*(\mathbf{X}\|\|\mathbf{Z})$	0.1453108
	$K_\alpha^*(\mathbf{X}\|\|\mathbf{W})$	0.2380369		$K_\alpha^*(\mathbf{X}\|\|\mathbf{W})$	0.2326783		$K_\alpha^*(\mathbf{X}\|\|\mathbf{W})$	0.1854185

6. CONCLUDING REMARKS

In this work, we have proposed multivariate fractional-order generalized CIMs. We have shown that the proposed measures are generalizations of some well-known inaccuracy measures. Some properties of the FGSCI measure have also been established. For brevity, results relating to the FGCI are not presented here, since they can be obtained in an analogous manner. In addition, we have proposed Jeffreys' inaccuracy measures between two (survival/distribution) copula functions and have studied some of their properties. The proposed measures have also been estimated using a semiparametric approach. Finally, a real data set has been considered, with which it has been demonstrated that the FGCI can be used effectively as a criteria to select the best model among a class of models.

One possible future work based on this work done is "to minimize the measure over a range of α parameter. This can be done for example using a discrete grid of values of it. Minimization of the measure will tell what is the nearness of one random vector is to another vector. Of course, one can even think of a bootstrap procedure after given a data to see whether such a infimum measure is significant of not."

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