REVSTAT – Statistical Journal
Volume 9, Number 3, November 2011, 213–226

RAYLEIGH DISTRIBUTION REVISITED VIA EXTENSION OF JEFFREYS PRIOR INFORMATION AND A NEW LOSS FUNCTION

Authors: Sanku Dey
– Department of Statistics, St. Anthony's College,
Shillong, Meghalaya, India
sanku_dey2k2003@yahoo.co.in

Tanujit Dey
– Department of Mathematics, The College of William & Mary,
Williamsburg, Virginia, USA
tdey@wm.edu

Received: November 2010 Revised: May 2011 Accepted: August 2011

Abstract:

- In this paper we present Bayes estimators of the parameter of the Rayleigh distribution, that stems from an extension of Jeffreys prior (Al-Kutubi (2005)) with a new loss function (Al-Bayyati (2002)). The performance of the proposed estimators has been compared in terms of bias and the mean squared error of the estimates based on Monte Carlo simulation study. We also derive the credible and the highest posterior density intervals for the Rayleigh parameter. We present an illustrative example to test how the Rayleigh distribution fits to a real data set.

Key-Words:

- *extension of Jeffreys prior; Jeffreys prior; Rayleigh distribution.*

AMS Subject Classification:

- 62C10, 62F10, 62F15, 65C10.

1. INTRODUCTION

The Rayleigh distribution has a wide range of applications including life testing experiments and clinical studies. One major application of this model is used in analyzing wind speed data. This distribution is a special case of the two parameter Weibull distribution with the shape parameter equal to 2. This statistical model was first introduced by Rayleigh (Rayleigh (1880)). Siddiqui (1962) discussed the origin and properties of the Rayleigh distribution. Several authors have contributed to this model, namely, Sinha and Howlader (1983), Ariyawansa and Templeton (1984), Howlader (1985), Howlader and Hossian (1995), Lalitha and Mishra (1996) and Abd Elfattah *et al.* (2006).

The probability distribution function (PDF) of one-parameter Rayleigh distribution is:

$$f(x|\sigma) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\,\sigma^2}\right), \qquad x \geq 0, \quad \sigma > 0 \; . \tag{1.1}$$

The objective of this article is to estimate the parameter and to obtain the credible and highest posterior density (HPD) intervals of the parameter of the Rayleigh distribution. We are proposing four different types of estimator. Under squared error loss function, there are two estimators formed by using Jeffreys prior and an extension of Jeffreys prior. The two remaining estimators are derived using the same Jeffreys prior and extension of Jeffreys prior under a new loss function introduced by Al-Bayyati (2002).

The article is organized as follows: Section 2 proposes two Bayes estimators of σ and the estimation is based on the squared error loss function using Jeffreys prior and an extension of Jeffreys prior information. Section 3 introduces the remaining two Bayes estimators of σ based on a loss function introduced by Al-Bayyati (2002) that uses Jeffreys and extension of Jeffreys prior. Section 4 presents the credible interval and the HPD interval for the Rayleigh parameter using extended Jeffreys prior. Section 5 is devoted to illustrative examples using both simulated and real life data sets, and Section 6 is the discussion.

2. PARAMETER ESTIMATION UNDER SQUARED ERROR LOSS FUNCTION

In this section, two different prior distributions are used for estimating the parameter of the Rayleigh distribution, namely; Jeffreys prior (Jeffreys (1961)) and extension of Jeffreys prior information.

2.1. Using Jeffreys prior

Considering there are n realizations, $\underline{x} = (x_1, x_2, ..., x_n)$ from (1.1). We consider Jeffreys prior as $g_1(\sigma) \propto \sqrt{I(\sigma)}$, where

$$I(\sigma) \;=\; -n\,E\left(\frac{\partial^2 \log f(x,\sigma)}{\partial\sigma^2}\right) \;=\; \frac{n}{\sigma^2}\;.$$

Then the joint p.d.f. is given by

$$f(\underline{x},\sigma) \;=\; \prod_{i=1}^{n} f(x_i,\sigma)\, g_1(\sigma)\;,$$

and the corresponding marginal PDF of $\underline{x}$ is obtained as

$$p(\underline{x}) \;=\; \int_0^\infty f(\underline{x},\sigma)\,d\sigma \;\propto\; \left[2^{n-1}\Gamma(n)\sqrt{n}\right]\frac{\prod_{i=1}^n x_i}{\left(\sum_{i=1}^n x_i^2\right)^n}\;.$$

The posterior PDF of σ has the following form

$$\pi_1(\sigma|\underline{x}) \;=\; \frac{2\left(\frac{s^2}{2}\right)^n}{\Gamma(n)\,\sigma^{2n+1}}\;\exp\!\left(-\frac{s^2}{2\,\sigma^2}\right), \tag{2.1}$$

where $s^2 = \sum_{i=1}^n x_i^2$. By using a squared error loss function $(L(\hat\sigma,\sigma) = c(\hat\sigma-\sigma)^2)$, for some constant c, the risk function is

$$\begin{aligned} R(\hat\sigma) \;&=\; \int_0^\infty L(\hat\sigma,\sigma)\,\pi_1(\sigma|\underline{x})\,d\sigma \\ &=\; c\,\hat\sigma^2 - 2\,c\,\frac{\Gamma\left(\frac{2n-1}{2}\right)}{\Gamma(n)}\sqrt{\frac{s^2}{2}}\;\hat\sigma + \frac{c}{(n-1)}\,\frac{s^2}{2}\;. \end{aligned}$$

The Bayes estimator $\hat\sigma_1$ is the solution of the equation $\frac{\partial R(\hat\sigma)}{\partial\hat\sigma} = 0$, which results in

$$\hat\sigma_1 \;=\; \frac{\Gamma\left(\frac{2n-1}{2}\right)}{\Gamma(n)}\left(\frac{s^2}{2}\right)^{1/2}. \tag{2.2}$$

2.2. Using extension of Jeffreys prior

Al-Kutubi (2005) proposed an extension of Jeffreys prior in the following form $g_2(\sigma) \propto (I(\sigma))^{c_1}$, $c_1 \in R^+$, where $I(\sigma)$ is the same as in Jeffreys prior. Moving along similar path, posterior PDF of σ has the following form:

$$\pi_2(\sigma|\underline{x}) \;=\; \frac{2\left(\frac{s^2}{2}\right)^{n+c_1-0.5}}{\Gamma(n+c_1-0.5)\,(\sigma^2)^{n+c_1}}\;\exp\!\left(-\frac{s^2}{2\sigma^2}\right)\;. \tag{2.3}$$

By using squared error loss function, we obtain the risk function as

$$R(\hat{\sigma}) = \int_0^\infty L(\hat{\sigma},\sigma)\,\pi_2(\sigma|\underline{x})\,d\sigma$$
$$= c\,\hat{\sigma}^2 - 2\,c\,\frac{\Gamma(n+c_1-1)}{\Gamma(n+c_1-0.5)}\sqrt{\frac{s^2}{2}}\,\hat{\sigma} + c\,\frac{\Gamma(n+c_1-1.5)}{\Gamma(n+c_1-0.5)}\cdot\frac{s^2}{2}\,.$$

The Bayes estimator $\hat{\sigma}_2$ is the solution of the equation $\frac{\partial R(\hat{\sigma})}{\partial\hat{\sigma}} = 0$, which results in

$$\hat{\sigma}_2 = \frac{\Gamma(n+c_1-1)}{\Gamma(n+c_1-0.5)}\left(\frac{s^2}{2}\right)^{1/2}. \tag{2.4}$$

Remark 2.1. Replacing $c_1 = 1/2$ in (2.4), the same Bayes estimator is obtained as in (2.2) corresponding to Jeffreys prior. By replacing $c_1 = 3/2$ in (2.4), Bayes estimator (2.4) becomes the estimator under Hartigan's prior (Hartigan (1964)).

3. PARAMETER ESTIMATION UNDER A NEW LOSS FUNCTION

This section uses a new loss function introduced by Al-Bayyati (2002). Employing this loss function, we obtain Bayes estimators using Jeffreys and extension of Jeffreys prior information.

Al-Bayyati (2002) introduced a new loss function of the form

$$L_A(\hat{\sigma},\sigma) = \sigma^{c_2}(\hat{\sigma}-\sigma)^2, \qquad c_2 \in R\,. \tag{3.1}$$

Here this loss function is used to obtain the estimator of the parameter of the Rayleigh distribution.

3.1. Using Jeffreys prior

By using the loss function in the form given in (3.1), we obtain the following risk function:

$$R(\hat{\sigma}) = \int_0^\infty L_A(\hat{\sigma},\sigma)\,\pi_1(\sigma|\underline{x})\,d\sigma$$
$$= \hat{\sigma}^2\,\frac{\Gamma\left(\frac{2n-c_2}{2}\right)}{\Gamma(n)}\left(\frac{s^2}{2}\right)^{\frac{c_2}{2}} - 2\,\hat{\sigma}\,\frac{\Gamma\left(\frac{2n-c_2-1}{2}\right)}{\Gamma(n)}\left(\frac{s^2}{2}\right)^{\frac{c_2+1}{2}} + \frac{\Gamma\left(\frac{2n-c_2-2}{2}\right)}{\Gamma(n)}\left(\frac{s^2}{2}\right)^{\frac{c_2+2}{2}}.$$

The Bayes estimator $\hat{\sigma}_3$ is the solution of the equation $\frac{\partial R(\hat{\sigma})}{\partial \hat{\sigma}} = 0$, which results in

$$\hat{\sigma}_3 = \frac{\Gamma\left(\frac{n-c_2-1}{2}\right)}{\Gamma\left(\frac{n-c_2}{2}\right)} \left(\frac{s^2}{2}\right)^{1/2} . \tag{3.2}$$

Remark 3.1. Replacing $c_2 = -2$ in (3.2), we get Bayes estimator under quadratic loss function (QLF) with Jeffreys prior, and if $c_2 = 0$ in (3.2), we get the Bayes estimator under squared error loss function with Jeffreys prior that reduces to (2.2).

3.2. Using extension of Jeffreys prior

Taking the posterior distribution (2.3) and the loss function in the form given in (3.1), the corresponding risk function becomes

$$\begin{aligned} R(\hat{\sigma}) &= \int_0^\infty L_A(\hat{\sigma}, \sigma)\, \pi_2(\sigma|\underline{x})\, d\sigma \\ &= \hat{\sigma}^2 \frac{\Gamma\left(\frac{2n+2c_1-c_2-1}{2}\right)}{\Gamma\left(\frac{2n+2c_1-1}{2}\right)} \left(\frac{s^2}{2}\right)^{\frac{c_2}{2}} - 2\,\hat{\sigma} \frac{\Gamma\left(\frac{2n+2c_1-c_2-2}{2}\right)}{\Gamma\left(\frac{2n+2c_1-1}{2}\right)} \left(\frac{s^2}{2}\right)^{\frac{c_2+1}{2}} \\ &\quad + \frac{\Gamma\left(\frac{2n+2c_1-c_2-3}{2}\right)}{\Gamma\left(\frac{2n+2c_1-1}{2}\right)} \left(\frac{s^2}{2}\right)^{\frac{c_2+2}{2}} . \end{aligned}$$

The Bayes estimator $\hat{\sigma}_4$ is the solution of the equation $\frac{\partial R(\hat{\sigma},\sigma)}{\partial \hat{\sigma}} = 0$, which results in

$$\hat{\sigma}_4 = \frac{\Gamma\left(\frac{2n+2c_1-c_2-2}{2}\right)}{\Gamma\left(\frac{2n+2c_1-c_2-1}{2}\right)} \left(\frac{s^2}{2}\right)^{1/2} . \tag{3.3}$$

Remark 3.2. Replacing $c_1 = 1/2$ and $c_2 = 0$ in (3.3), we get the Bayes estimator under squared error loss function with Jeffreys prior which is same as (2.2) and if $c_1 = 1/2$ and $c_2 = -2$ in (3.3), we get the Bayes estimator under QLF with Jeffreys prior.

4. THE CREDIBLE INTERVAL AND THE HPD INTERVAL USING EXTENDED JEFFREYS PRIOR

Earlier we derived the Bayesian point estimator of the unknown parameter, but it is important to account for posterior uncertainty. The purpose of this section is to derive the credible interval and HPD interval for the Rayleigh parameter under extended Jeffreys prior. First, we will construct the credible interval and then we will introduce the HPD interval.

From the expression (2.3), we see that $\frac{2s^2}{\sigma^2}$ follows a Chi-Squared distribution with $2(n+c_1-0.5)$ degrees of freedom $\left[\chi^2_{(2(n+c_1-0.5))}\right]$. So, to construct a $100(1-\alpha)\%$ credible interval for σ, we have

$$\begin{aligned} 1-\alpha &= P\left[\chi^2_{\left(1-\frac{\alpha}{2},\,2(n+c_1-0.5)\right)} < \frac{2\,s^2}{\sigma^2} < \chi^2_{\left(\frac{\alpha}{2},\,2(n+c_1-0.5)\right)}\right] \\ &= P\left[\frac{2\,s^2}{\chi^2_{\left(\frac{\alpha}{2},\,2(n+c_1-0.5)\right)}} < \sigma^2 < \frac{2\,s^2}{\chi^2_{\left(1-\frac{\alpha}{2},\,2(n+c_1-0.5)\right)}}\right]. \end{aligned}$$

Therefore, we get the $100(1-\alpha)\%$ credible interval for σ as

$$\left[C_L(\sigma),\, C_U(\sigma)\right] = \left[\sqrt{\frac{2\,s^2}{\chi^2_{\left(\frac{\alpha}{2},\,2(n+c_1-0.5)\right)}}}\,,\ \sqrt{\frac{2\,s^2}{\chi^2_{\left(1-\frac{\alpha}{2},\,2(n+c_1-0.5)\right)}}}\right]. \tag{4.1}$$

The HPD interval is one of the most effective tool that helps to measure posterior uncertainty. As discussed in Box and Tiao (1973), a HPD interval is such that the posterior density for every point inside the interval is greater than that for every point outside it, so that the intervals include the more probable values of the parameter. For a given probability, say $1-\alpha$; the HPD interval is of the shortest interval to offer a pertinent summary of the posterior knowledge of the parameter.

Since the PDF (2.3) is unimodal, the HPD interval (H_1, H_2) with probability $1-\alpha$, for σ must satisfy the equations (4.2) and (4.3) simultaneously (see Box and Tiao (1973)).

The $100(1-\alpha)\%$ HPD interval $[H_1, H_2]$ for σ is derived from the following equations:

$$\int_{H_1}^{H_2} \pi_2(\sigma|\underline{x})\, d\sigma = 1-\alpha \tag{4.2}$$

and

$$\pi_2(H_1|\underline{x}) = \pi_2(H_2|\underline{x})\ . \tag{4.3}$$

After simplification, the equations (4.2) and (4.3) take the following form:

$$\int_{\frac{s^2}{2H_2^2}}^{\frac{s^2}{2H_1^2}} \frac{1}{\Gamma(n+c_1-0.5)}\, z^{n+c_1-1.5} e^{-z}\, dz = 1-\alpha \tag{4.4}$$

and

$$\left(\frac{H_2}{H_1}\right)^{2n+2c_1} = \exp\left[\frac{s^2}{2H_1^2} - \frac{s^2}{2H_2^2}\right]. \tag{4.5}$$

The HPD interval $[H_1, H_2]$ is the simultaneous solution of (4.4) and (4.5).

5. ILLUSTRATIVE EXAMPLES

This section presents the performance of four proposed estimators based on a simulation study and real life data application.

5.1. Simulation study

In this section, we carry out a Monte Carlo simulation to study the performance of the proposed Bayes estimators. The performance is evaluated based on the bias and mean squared error (MSE) criteria for different sample sizes ($n = 10, 20, 30$) and for different prior parameters. In computing the estimators, we have generated samples from (1.1) with $\sigma = 0.5$ and 1, and repeated the process for 10,000 times. The average bias and MSE's are presented in Tables 1 and 2, respectively. In our simulation study, we have used $c_1 = 0.5, 1.0, 1.5, 2.0$ and $c_2 = \pm 1, \pm 2$.

MSE of $\hat{\sigma}$ is defined by $MSE(\hat{\sigma}) = E(\hat{\sigma} - \sigma)^2 = \mathrm{Var}(\hat{\sigma}) + [\mathrm{Bias}(\hat{\sigma})]^2$. Note that 10,000 repetitions will provide accuracy in the order $\pm(10\,000)^{-0.5} = \pm 0.01$ (Karian and Dudewicz (1999)), so results are reported to four decimal places.

Graphical depiction of data is often times a better representation of results. The goal is to graphically present similar results to offer a thorough assessment of the four estimators corresponding to their biases and MSE's. Results in Figure 1 is obtained from a simulation study. Herein we sampled data from (1.1) with $\sigma = 1$ with five different sample sizes ($n = 10, 20, 30, 40, 50$). Four estimators are calculated based on these samples with the values of $c_1 = 0.5, 1.0, 1.5, 2.0$ and $c_2 = \pm 1, \pm 2$.

Figure 1 is a conditional plot for biases and MSE's of four estimators, conditioned by sample sizes obtained from the simulated study. In Figure 1, we see that for $\hat{\sigma}_1$, bias and MSE are very consistent, irrespective of sample size and both are approaching zero as sample size increases; whereas for the remaining estimators, they are out of sync for different choices of c_1 and c_2. When sample size increases from 10 to 50, the bias and MSE both decreases quite significantly.

Table 1: Bias and mean squared error (MSE) of four different estimators. Results are based on 10,000 simulations from (1.1) with $\sigma = 0.5$.

n	Criteria	$\hat{\sigma}_1$				$\hat{\sigma}_2$				$\hat{\sigma}_3$				$\hat{\sigma}_4$			
		c_2				c_2				c_2				c_2			
		−1	1	−2	2	−1	1	−2	2	−1	1	−2	2	−1	1	−2	2
$c_1 = 0.5$																	
10	Bias	0.0139	0.0133	0.0128	0.0144	0.0139	0.0133	0.0128	0.0144	0.0005	0.0277	−0.0129	0.0446	0.0005	0.0277	−0.0129	0.0446
	MSE	0.0068	0.0069	0.0068	0.0069	0.0068	0.0069	0.0068	0.0069	0.0063	0.0079	0.0061	0.0095	0.0063	0.0079	0.0061	0.0095
20	Bias	0.0070	0.0070	0.0058	0.0059	0.0070	0.0070	0.0058	0.0059	0.0006	0.0137	−0.0068	0.0196	0.0006	0.0137	−0.0068	0.0196
	MSE	0.0033	0.0033	0.0033	0.0032	0.0033	0.0033	0.0033	0.0032	0.0031	0.0035	0.0031	0.0038	0.0031	0.0035	0.0031	0.0038
30	Bias	0.0040	0.0048	0.0042	0.0036	0.0040	0.0048	0.0042	0.0036	−0.0002	0.0092	−0.0042	0.0125	−0.0002	0.0092	−0.0042	0.0125
	MSE	0.0021	0.0021	0.0022	0.0022	0.0021	0.0021	0.0022	0.0022	0.0021	0.0022	0.0021	0.0024	0.0021	0.0022	0.0021	0.0024
$c_1 = 1.0$																	
10	Bias	0.0144	0.0124	0.0130	0.0132	0.0010	−0.0009	−0.0003	−0.0001	0.0010	0.0268	−0.0126	0.0434	−0.0113	0.0124	−0.0241	0.0276
	MSE	0.0069	0.0068	0.0068	0.0067	0.0063	0.0063	0.0063	0.0062	0.0063	0.0078	0.0061	0.0092	0.0061	0.0068	0.0063	0.0077
20	Bias	0.0072	0.0057	0.0072	0.0072	0.0007	−0.0007	0.0008	0.0008	0.0007	0.0124	−0.0054	0.0209	−0.0055	0.0057	−0.0114	0.0139
	MSE	0.0033	0.0032	0.0033	0.0033	0.0032	0.0031	0.0032	0.0032	0.0032	0.0034	0.0031	0.0039	0.0032	0.0032	0.0032	0.0036
30	Bias	0.0047	0.0038	0.0046	0.0051	0.0004	−0.0004	0.0004	0.0008	0.0004	0.0082	−0.0038	0.0139	−0.0038	0.0038	−0.0078	0.0094
	MSE	0.0022	0.0021	0.0021	0.0022	0.0021	0.0021	0.0021	0.0021	0.0021	0.0022	0.0021	0.0024	0.0021	0.0021	0.0021	0.0023
$c_1 = 1.5$																	
10	Bias	0.0148	0.0134	0.0131	0.0116	−0.0109	−0.0122	−0.0125	−0.0140	0.0014	0.0279	−0.0125	0.0416	−0.0224	0.0001	−0.0347	0.0116
	MSE	0.0069	0.0068	0.0070	0.0067	0.0061	0.0062	0.0063	0.0061	0.0063	0.0078	0.0063	0.0091	0.0062	0.0063	0.0068	0.0067
20	Bias	0.0069	0.0069	0.0075	0.0057	−0.0058	−0.0058	−0.0052	−0.0070	0.0004	0.0136	−0.0052	0.0193	−0.0118	0.0004	−0.0170	0.0057
	MSE	0.0034	0.0033	0.0033	0.0031	0.0032	0.0031	0.0031	0.0030	0.0032	0.0035	0.0031	0.0036	0.0032	0.0032	0.0032	0.0031
30	Bias	0.0049	0.0044	0.0041	0.0046	−0.0035	−0.0040	−0.0043	−0.0038	0.0006	0.0088	−0.0043	0.0134	−0.0076	0.0002	−0.0123	0.0046
	MSE	0.0022	0.0022	0.0021	0.0022	0.0021	0.0021	0.0020	0.0021	0.0021	0.0023	0.0020	0.0024	0.0021	0.0021	0.0021	0.0022
$c_1 = 2.0$																	
10	Bias	0.0138	0.0145	0.0145	0.0142	−0.0234	−0.0228	−0.0228	−0.0230	0.0004	0.0289	−0.0113	0.0445	−0.0341	−0.0113	−0.0435	0.0009
	MSE	0.0068	0.0071	0.0069	0.0069	0.0062	0.0064	0.0063	0.0063	0.0062	0.0081	0.0061	0.0095	0.0066	0.0063	0.0071	0.0064
20	Bias	0.0068	0.0057	0.0062	0.0067	−0.0119	−0.0130	−0.0124	−0.0120	0.0003	0.0124	−0.0064	0.0204	−0.0177	−0.0070	−0.0238	0.0002
	MSE	0.0033	0.0033	0.0033	0.0033	0.0031	0.0032	0.0032	0.0031	0.0032	0.0035	0.0031	0.0038	0.0032	0.0032	0.0034	0.0031
30	Bias	0.0041	0.0047	0.0038	0.0035	−0.0083	−0.0078	−0.0086	−0.0090	−0.0001	0.0091	−0.0046	0.0123	−0.0123	−0.0037	−0.0164	−0.0008
	MSE	0.0021	0.0022	0.0021	0.0022	0.0021	0.0021	0.0021	0.0021	0.0021	0.0023	0.0021	0.0024	0.0021	0.0021	0.0022	0.0021

Table 2: Bias and mean squared error (MSE) of four different estimators. Results are based on 10,000 simulations from (1.1) with $\sigma = 1$.

n	Criteria	$\hat{\sigma}_1$				$\hat{\sigma}_2$				$\hat{\sigma}_3$				$\hat{\sigma}_4$			
		c_2				c_2				c_2				c_2			
		−1	1	−2	2	−1	1	−2	2	−1	1	−2	2	−1	1	−2	2
$c_1 = 0.5$																	
10	Bias	0.0263	0.0237	0.0279	0.0264	0.0263	0.0237	0.0279	0.0264	−0.0003	0.0525	−0.0235	0.0868	−0.0003	0.0525	−0.0235	0.0868
	MSE	0.0268	0.0271	0.0275	0.0269	0.0268	0.0271	0.0275	0.0269	0.0247	0.0309	0.0247	0.0369	0.0247	0.0309	0.0247	0.0369
20	Bias	0.0132	0.0137	0.0145	0.0132	0.0132	0.0137	0.0145	0.0132	0.0003	0.0271	−0.0108	0.0406	0.0003	0.0271	−0.0108	0.0406
	MSE	0.0130	0.0132	0.0132	0.0132	0.0130	0.0132	0.0132	0.0132	0.0125	0.0141	0.0125	0.0154	0.0125	0.0141	0.0125	0.0154
30	Bias	0.0069	0.0076	0.0074	0.0092	0.0069	0.0076	0.0074	0.0092	−0.0016	0.0164	−0.0094	0.0269	−0.0016	0.0164	−0.0094	0.0269
	MSE	0.0086	0.0085	0.0085	0.0086	0.0086	0.0085	0.0085	0.0086	0.0084	0.0085	0.0083	0.0095	0.0084	0.0085	0.0083	0.0095
$c_1 = 1.0$																	
10	Bias	0.0278	0.0280	0.0237	0.0263	0.0011	0.0014	−0.0029	−0.0003	0.0011	0.0570	−0.0275	0.0867	−0.0236	0.0280	−0.0504	0.0552
	MSE	0.0278	0.0264	0.0275	0.0271	0.0256	0.0243	0.0255	0.0251	0.0256	0.0304	0.0251	0.0372	0.0249	0.0264	0.0257	0.0310
20	Bias	0.0113	0.0126	0.0131	0.0133	−0.0016	−0.0003	0.0002	0.0004	−0.0016	0.0260	−0.0122	0.0407	−0.0140	0.0126	−0.0242	0.0267
	MSE	0.0128	0.0131	0.0129	0.0134	0.0123	0.0126	0.0124	0.0129	0.0123	0.0140	0.0123	0.0156	0.0122	0.0131	0.0124	0.0143
30	Bias	0.0072	0.0083	0.0083	0.0080	−0.0013	−0.0002	−0.0002	−0.0005	−0.0013	0.0170	−0.0085	0.0257	−0.0096	0.0083	−0.0166	0.0167
	MSE	0.0086	0.0087	0.0086	0.0085	0.0084	0.0084	0.0084	0.0083	0.0084	0.0090	0.0084	0.0094	0.0083	0.0087	0.0084	0.0089
$c_1 = 1.5$																	
10	Bias	0.0273	0.0244	0.0280	0.0256	−0.0241	−0.0268	−0.0234	−0.0257	0.0006	0.0533	−0.0234	0.0859	−0.0470	−0.0022	−0.0678	0.0256
	MSE	0.0270	0.0277	0.0278	0.0275	0.0243	0.0252	0.0249	0.0249	0.0249	0.0315	0.0249	0.0374	0.0248	0.0257	0.0268	0.0275
20	Bias	0.0123	0.0132	0.0114	0.0128	−0.0130	−0.0122	−0.0139	−0.0126	−0.0006	0.0266	−0.0139	0.0401	−0.0249	0.0002	−0.0373	0.0128
	MSE	0.0133	0.0132	0.0129	0.0131	0.0127	0.0126	0.0124	0.0125	0.0128	0.0141	0.0124	0.0153	0.0128	0.0127	0.0130	0.0131
30	Bias	0.0091	0.0096	0.0064	0.0090	−0.0077	−0.0072	−0.0104	−0.0078	0.0006	0.0183	−0.0104	0.0267	−0.0158	0.0011	−0.0263	0.0090
	MSE	0.0087	0.0085	0.0086	0.0086	0.0084	0.0082	0.0084	0.0083	0.0085	0.0089	0.0084	0.0095	0.0085	0.0082	0.0087	0.0086
$c_1 = 2.0$																	
10	Bias	0.0289	0.0251	0.0263	0.0288	−0.0455	−0.0491	−0.0479	−0.0456	0.0022	0.0539	−0.0250	0.0893	−0.0669	−0.0262	−0.0893	0.0021
	MSE	0.0274	0.0273	0.0269	0.0267	0.0249	0.0253	0.0249	0.0244	0.0252	0.0311	0.0243	0.0370	0.0263	0.0247	0.0286	0.0246
20	Bias	0.0126	0.0121	0.0134	0.0116	−0.0247	−0.0252	−0.0239	−0.0256	−0.0003	0.0255	−0.0119	0.0390	−0.0362	−0.0132	−0.0466	−0.0013
	MSE	0.0133	0.0130	0.0130	0.0129	0.0128	0.0126	0.0125	0.0125	0.0128	0.0139	0.0124	0.0150	0.0132	0.0124	0.0135	0.0125
30	Bias	0.0077	0.0089	0.0087	0.0088	−0.0172	−0.0160	−0.0162	−0.0161	−0.0008	0.0177	−0.0081	0.0265	−0.0251	−0.0079	−0.0318	0.0003
	MSE	0.0086	0.0085	0.0084	0.0088	0.0084	0.0083	0.0082	0.0086	0.0084	0.0089	0.0082	0.0097	0.0086	0.0082	0.0087	0.0086

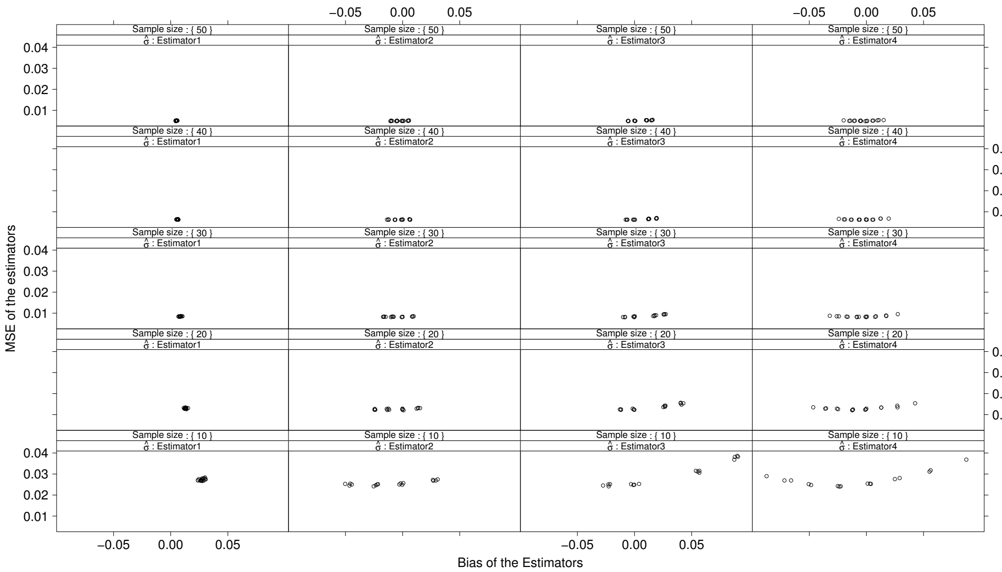

Figure 1: Conditional plot of bias and MSE of four estimators, conditioned on sample size from a simulated study with $\sigma = 1$. In the plot "Estimator1" stands for $\hat{\sigma}_1$, "Estimator2" for $\hat{\sigma}_2$, "Estimator3" for $\hat{\sigma}_3$ and "Estimator4" for $\hat{\sigma}_4$, respectively. Results are based on 10,000 simulations.

5.2. A real life data example

Here we consider an example of a real life data set for comparing the performances of four estimators with the maximum likelihood estimator (MLE) of Rayleigh distribution. Based on the model (1.1), the MLE of σ is given by $\hat{\sigma}_{MLE} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} x_i^2}$. We make use of a wind speed data set (Albuhairi (2006)) of Taiz, located southwest of Yemen. Average monthly wind speed for the year 2002 has been used for this analysis. Before performing estimation of parameter, we have checked goodness of fit of this data by using three different measures: Kolmogorov–Smirnov (KS) test, Anderson–Darling (AD) test and χ^2 goodness of fit test. KS test (test statistic value = 0.35711 with p-value 0.07098), AD test (test statistic = 1.9879) and χ^2 test (test statistic value = 0.8251 with p-value = 0.36369) suggest that one-parameter Rayleigh provides an adequate fit to this data set. Based on this 12 data points, we find $\hat{\sigma}_{MLE} = 3.1593$. Table 3 presents the values of four estimators with choices of $c_1 = 0.5, 1.0, 1.5, 2.0$ and $c_2 = \pm 1.0, \pm 2.0$.

An important issue is to determine whether these Bayes estimators give better estimates than the MLE. To test this, we have computed Kolmogorov–Smirnov (KS) distances between the empirical distribution and the fitted distribution functions for MLE and other Bayes estimators. In all cases, the KS distance for Bayes estimators are smaller than the distance using MLE (results are not reported here).

Table 3: Four different estimators of the parameter σ based on wind speed data with values of $c_1 = 0.5, 1.0, 1.5, 2.0$ and $c_2 = \pm 1, \pm 2$.

c_1	c_2	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\hat{\sigma}_3$	$\hat{\sigma}_4$
0.5	−1	3.1946	3.1946	3.1779	3.1779
	1	3.1946	3.1946	3.2117	3.2117
	−2	3.1946	3.1946	3.1614	3.1614
	2	3.1946	3.1946	3.2290	3.2290
1.0	−1	3.1946	3.1779	3.1779	3.1614
	1	3.1946	3.1779	3.2117	3.1946
	−2	3.1946	3.1779	3.1614	3.1451
	2	3.1946	3.1779	3.2290	3.2117
1.5	−1	3.1946	3.1614	3.1779	3.1451
	1	3.1946	3.1614	3.2117	3.1779
	−2	3.1946	3.1614	3.1614	3.1291
	2	3.1946	3.1614	3.2290	3.1946
2.0	−1	3.1946	3.1451	3.1779	3.1291
	1	3.1946	3.1451	3.2117	3.1614
	−2	3.1946	3.1451	3.1614	3.1133
	2	3.1946	3.1451	3.2290	3.1779

Table 4 presents the 95% credible and the HPD intervals for σ under extended Jeffreys prior distribution. For comparison, we have calculated 95% confidence interval using the asymptotic variance of the MLE as $(2.2655, 4.0531)$. The width of the HPD intervals are smaller than the width of the confidence interval, corresponding to all choices of c_1 values, whereas the 95% credible intervals provide larger width compared to the HPD intervals and the confidence interval.

Table 4: The 95% Credible intervals and HPD intervals for wind speed data.

Intervals	c_1			
	0.5	1.0	1.5	2.0
Credible	(3.4887, 6.2155)	(3.4332, 6.0429)	(3.3805, 5.8827)	(3.3304, 5.7336)
HPD	(2.4909, 4.2542)	(2.3664, 4.1413)	(2.3134, 4.0362)	(2.2819, 3.9381)

6. DISCUSSION

From the simulation study, we establish that the estimators are asymptotically unbiased and consistent. For moderate or large sample sizes, all the estimators with Hartigan's prior along with QLF have minimal biases. We also notice that, except $\hat{\sigma}_1$, other estimators underestimate when $c_1 = 1.5$ and $c_2 = -2$. When we take into account Jeffreys prior with QLF, $\hat{\sigma}_3$ and $\hat{\sigma}_4$ underestimates, whereas $\hat{\sigma}_1$ and $\hat{\sigma}_2$ overestimates. Finally, when comparing the functioning of all the estimators, we illustrate that as far as biases are concerned, $\hat{\sigma}_2$ performs better than $\hat{\sigma}_1$ in view of Hartigan's prior. Using KS distance we find that four Bayes estimators provide convincingly better estimates of σ than $\hat{\sigma}_{MLE}$ based on wind speed data.

ACKNOWLEDGMENTS

The authors would like to thank the referees and the Editor and the Associate editor who have helped to improve the paper substantially.

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